Sustainability Analysis for a Forestry Management Model

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Aim of this work :

Determine **analytically**, on a **simple** model of forest evolution, when a **sustainable** harvesting is possible.

A simple model of forest evolution (I)

Assumptions :

A1 : The forest is divided in S parcels.
On each parcel, trees have the same age.

A2 : Only parcels with mature trees are allowed to be harvested.

A3 : When a parcel is harvested, all trees are cut and replaced by young ones.

A4 : Spatial interactions are neglected.

State variable : Let $n \in \mathbb{N}$ be the mature age,

$$x(t) = \begin{bmatrix} x_n(t) \\ \vdots \\ x_1(t) \end{bmatrix} \qquad \begin{aligned} x_n = & \text{ nb of cells with mature trees,} \\ x_i = & \text{ nb of cells whose tree age } \in (i, i+1] \end{aligned}$$

A simple model of forest evolution (II)

$$x(t+1) = Ax(t) + Bh(t)$$

Hypotheses :

H1 : $A_{i,j} \in [0,1],\ \forall i,j \qquad \sum_{k=1}^{n} A_{k,j} = 1,\ \forall j$

H2 : $B = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

H3 : $h(t) \in [0, Cx(t)]$ where $C = [\,1\,0\,\ldots\,0\,]$

Lemma : Under H1-H2-H3, the system has an invariant :

$$\sum_{i=1}^{n} x_i(t) = Cte = S, \quad \forall t \geq 0$$

Examples

1. The eternal population :

$$A = \begin{bmatrix} 1 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

2. The almost eternal population :

$$A_\alpha = \begin{bmatrix} 1-\alpha & 1-\alpha & 0 & & 0 \\ & & 1-\alpha & \ddots & \vdots \\ & & & \ddots & 0 \\ & & & & 1-\alpha \\ \alpha & \cdots & \cdots & \cdots & \alpha \end{bmatrix} \qquad \alpha \in (0,1).$$

Sustainability

Definitions :

1. **Sustainable harvesting** : an admissible policy $h(0), h(1), \cdots$ that ensures a minimal cut every year $t \in [0, +\infty)$.
2. **Sustainable harvesting of level** $\underline{h}$: an admissible policy $h(.)$ such that $h(t) \geq \underline{h}$, $\forall t \geq 0$.
3. **Viability kernel** $\mathrm{Viab}_{\underline{h}}$: the set of all initial conditions $x(0)$ from which there exists at least a sustainable harvesting of level $\underline{h}$.
4. **Maximal sustainable level** $\underline{h}^*$: the largest value $\underline{h}$ such that $\mathrm{Viab}_{\underline{h}} \neq \emptyset$.
5. **Sustainable regulation h[ ]** : a feedback law $x \in \mathrm{Viab}_{\underline{h}} \longrightarrow h[x]$ that ensures sustainability from any initial condition in $\mathrm{Viab}_{\underline{h}}$.

Remarks :

1. This is not an optimization problem (no utility function, no price). There is no discount factor.
2. Myopic strategies are known to be, in general, non sustainable, such as the greedy one that leads to non-sustainable cycles.

The curse of dimensionality

With the eternal population model,

$$\begin{array}{l} x(0) \in \mathbb{N}^n \\ h(t) \in \mathbb{N},\ \forall t \geq 0 \end{array} \implies x(t) \in \mathbb{N}^n$$

and

$$\mathrm{card}\left\{ x \in \mathbb{N}^n \mid \sum_{i=1}^{n} x_i = S \right\} = C_{S+n-1}^{S}$$

Example : 100 parcels, 100 years as mature age
$\Rightarrow$ card $\simeq 10^{60}$.

Results

$$\text{Define } T = \begin{bmatrix} 0 & & 0 \\ & \ddots & \\ 1 & & 0 \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \quad \mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

Proposition 1 :

$$\begin{cases} O \text{ full rank} \\ \mathbb{1}' O^{-1} \geq 0 \\ OB \leq 0 \end{cases} \Rightarrow \underline{h}^* = \frac{S}{\mathbb{1}' O^{-1} (\mathbb{1} - TOB)}$$

Proposition 2 :

$$\begin{cases} CA^n = CA^{n-1} \\ CA^k B \leq 0, \ k = 1 \cdots n-2 \\ CA^{n-1} B = 0 \end{cases} \Rightarrow$$

$$\text{Viab}_{\underline{h}} = \{x \in \mathbb{R}^n_+ \,|\, Ox \geq (\mathbb{1} - TOB)\underline{h}\}$$

Proposition 3 : $\underline{h}$-sustainable regulations are given by :

$$h[x] \in \{h \in [0, Cx] \text{ s.t. } OBh \geq (\mathbb{1} - TOB)\underline{h} - OAx\}$$

Simulations I
(eternal population)

$$S = 20, \quad n = 4 \quad \Rightarrow \quad \underline{h}^* = 5$$

Maximal sustainable harvesting policy ($\underline{h} = 4$) :

t	0	1	2	3	4	5	6	7	8	9	10
x_5	2.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
x_4	6.0	5.0	3.0	4.0	8.0	4.0	4.0	4.0	8.0	4.0	4.0
x_3	5.0	3.0	4.0	8.0	4.0	4.0	4.0	8.0	4.0	4.0	4.0
x_2	3.0	4.0	8.0	4.0	4.0	4.0	8.0	4.0	4.0	4.0	8.0
x_1	4.0	8.0	4.0	4.0	4.0	8.0	4.0	4.0	4.0	8.0	4.0
h_M	8.0	4.0	4.0	4.0	8.0	4.0	4.0	4.0	8.0	4.0	4.0

Inertial sustainable harvesting policy ($\underline{h} = 4$) :

t	0	1	2	3	4	5	6	7	8	9	10
x_5	2.0	3.5	4.0	2.5	2.0	2.0	2.0	2.0	2.0	2.0	2.0
x_4	6.0	5.0	3.0	4.0	4.5	4.5	4.5	4.5	4.5	4.5	4.5
x_3	5.0	3.0	4.0	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5
x_2	3.0	4.0	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5
x_1	4.0	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5
h_I	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5

Simulations II
(almost eternal population)

$$S = 20, \quad n = 5, \quad \alpha = 0.07 \quad \Rightarrow \quad \underline{h}^* \simeq 4.2$$

Maximal sustainable harvesting policy ($\underline{h} = 4$) :

t	0	1	2	3	4	5	6	7	8	9	10	$\cdots$	$+\infty$
x_5	2.0	2.7	2.9	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	$\rightarrow$	0.0
x_4	6.0	4.6	2.6	3.2	5.0	4.3	4.3	4.3	4.8	4.4	4.4	$\rightarrow$	4.5
x_3	5.0	2.8	3.5	5.3	4.7	4.7	4.7	5.2	4.7	4.7	4.7	$\rightarrow$	4.8
x_2	3.0	3.7	5.7	5.0	5.0	5.0	5.5	5.0	5.0	5.0	5.5	$\rightarrow$	5.2
x_1	4.0	6.1	5.4	5.4	5.4	6.0	5.4	5.4	5.4	5.9	5.5	$\rightarrow$	5.6
h_M	4.7	4.0	4.0	4.0	4.6	4.0	4.0	4.0	4.5	4.0	4.0	$\rightarrow$	4.2

Inertial sustainable harvesting policy ($\underline{h} = 4$) :

t	0	1	2	3	4	5	6	7	8	9	10
x_5	2.0	3.0	2.9	1.0	0.0	0.41	0.56	0.56	0.56	0.56	0.56
x_4	6.0	4.6	2.6	3.2	4.7	4.5	4.3	4.3	4.3	4.3	4.3
x_3	5.0	2.8	3.5	5.1	4.8	4.7	4.7	4.7	4.7	4.7	4.7
x_2	3.0	3.7	5.5	5.2	5.0	5.0	5.0	5.0	5.0	5.0	5.0
x_1	4.0	6.0	5.6	5.4	5.4	5.4	5.4	5.4	5.4	5.4	5.4
h_I	4.5	4.2	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0