
Chapter VIII. Optimal replacement in the dairy herd: A multi-component system (p 105-130)

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Optimal Replacement in the Dairy Herd: A Multi-component System

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ABSTRACT

The dairy herd is described as a multi-component system, where the components are the cows and heifers. The problem of finding an optimal replacement policy to the multi-component system is considered. The complication of the multi-component model is that, if the supply of heifers is limited (i.e. the dairy farmer uses only home-grown heifers), the replacement decision concerning a cow does not only depend on the state of that particular cow but also on the states of the other cows and heifers in the herd. Initially, it is demonstrated that the multi-component replacement problem may be formulated as an ordinary Markov decision process. Unfortunately, the model is far too large to be solved by any known methods. Therefore, an approximate method combining dynamic programming and stochastic simulation in the determination of a set of descriptive parameters is suggested. The parameters are used in the calculation of the multi-component replacement criterion for cows as well as for heifers. The method has been tested by extensive simulations under 100 different conditions concerning prices and average milk yield of the herd. It was concluded that, when the replacement costs (the price of a heifer minus the price of a calf and the carcass value of a cow) are small, the method improves the economic results considerably compared to the usual models, assuming an unlimited supply of heifers. The information concerning heifers, which is provided by the method, makes it relevant even in cases where the replacement costs are large. The basic idea of the study may be relevant in a more general range of problems involving replacement under some constraint.

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1 INTRODUCTION

The aim of this study was to develop a method for finding an approximate solution for the optimal replacement policy in the dairy herd regarded as a multi-component system. In the literature, several studies deal with optimal dairy cow replacement regarded as a single-component system (i.e. only one cow is considered at a time, assuming an unlimited supply of heifers for replacement). A review of such studies is given by Arendonk (1984), and more recent examples are Arendonk and Dijkstraizen (1985), Arendonk (1986) and Kristensen (1987, 1989). The latter models are very detailed and, from a theoretical point of view, the single-component replacement problem in dairy herds may be regarded as having been solved to a satisfactory degree.

Most replacement studies in the literature deal with a single-component system (i.e. only one producing unit (component) is considered, assuming an unlimited supply of replacement units). In a multi-component system, however, several producing units (components) are considered simultaneously. If only the number of components in operation is limited, but the supply of replacements is unlimited, the problem is identical to that of the single-component model, because the decision to replace a component does not influence the possibility of replacing any of the other components. On the other hand, if the supply of replacements is limited, the replacement of one component will decrease the possibility of replacing others, because the number of replacements may not suffice. Therefore, the replacement decision concerning one component does not only depend on the state of that particular component but also on those of the other components of the system. Instead of a relatively simple comparison of two components (the one in operation and the replacement) the problem is now faced of choosing the optimal composition of components from the available population of the components in operation and the available replacements.

It is assumed that the interaction is due to technical and economic dependency so that each component is assumed stochastically independent of the others. If the dairy herd is regarded as a multi-component system, the system is the entire herd, whereas the components in operation are the individual cows and the replacements are the available heifers.

A special feature of the dairy herd replacement problem, compared to a general formulation of the multi-component replacement problem, is that the supply of replacements is not exogenous to the process but is actually generated by the process itself. Since many dairy farmers only use home-grown heifers as replacements (mainly because of the risk of introducing infectious diseases in the herd when heifers are bought at the market), the supply of heifers will be limited to those born in the herd. This further raises
the problem of deciding how many female calves to raise as potential replacements, which is also briefly discussed in this paper.

Most single-component studies in the literature dealing with dairy cows use dynamic programming and Markov decision processes in the determination of optimal replacement policies. Some of the models are very large. Thus, Arendonk and Dijkstraizen (1985) used a model with 174,000 states, reported by Arendonk (1988), and the model of Kristensen (1989) contained 180,000 states. Also, the multi-component problem can be formulated as a Markov decision process, but, since the states of all components should be considered simultaneously, the size of the total model will be far beyond computational capacity. Therefore, the need for approximate methods arises.

Ben-Ari and Gal (1986) discussed this problem and introduced a method called 'Parameter Iteration'. The idea is to approximate the total expected profit of the herd at a given composition by a function involving a set of parameters describing the relations between the total expected profit and the present herd composition. The parameters are determined in each situation by an iterative method.

Also, the method described in this paper is based on a parameter iteration technique, but the implementation of the idea is fundamentally different from that of Ben-Ari and Gal (1986). The main reasons for dealing with the problem again are as follows:

1. The approximation used by Ben-Ari and Gal (1986) is exact when the supply of heifers is unlimited (i.e. the opposite situation of the one studied).
2. The evaluation of the method seems insufficient in Ben-Ari and Gal (1986). No results have been presented showing the benefits of the method over the results from the underlying single-component model.
3. The single-component model of Ben-Ari and Gal (1986) was very simple, containing only 180 states.

In the present study, the problem is discussed under the assumption that no milk quota is present. An additional paper discussing the multi-component replacement problem under a milk quota will be considered later.

2 THE DAIRY HERD AS A MULTI-COMPONENT SYSTEM

In this section, the multi-component replacement problem in dairy herds is described and it is shown that, in principle, it may be formulated as an ordinary Markov decision process with known parameters.

A dairy herd with a limited maximum number of cows ($L$) was considered.
Each cow and its successors are represented by a Markov decision process with known action and state spaces and known parameters (the single-component model). The reward of state \( i \) under the action \( a \) (\( a = \text{'keep'} \) or \( \text{'replace'} \)) is denoted as \( r_i^a \) and the transition probability from state \( i \) at the present stage to state \( j \) at the following stage under action \( a \) is denoted as \( p_{ij}^a \). The state \( i \) of a cow reflects the important characteristics of the cow (i.e. milk yield, age, reproductive status, etc.). Also, a state representing the absence of a cow (an empty stall) must be included.

The Markov decision process may be an ordinary process, as described by Howard (1960), or a Hierarchic Markov process, as described by Kristensen (1988). In both cases, there are iterative methods to determine an optimal policy under infinite horizon (i.e. an infinite number of stages) assuming an unlimited supply of heifers. Under the optimal policy the present value (\( f_i \)) can be calculated, the total expected discounted rewards of the process starting in state \( i \) and running over an infinite number of stages following an optimal policy.

The single-component model of this study is identical to the model of Kristensen (1989). The state variables of that model are the genetic merit (5 classes), the lactation number (6 classes), the stage of lactation (18 classes), the milk yield of previous and present lactation (15 classes each), and the pregnancy status (8 classes). The model is of the hierarchic type with a total of 180,000 states.

A Markov decision process describing the entire multi-component system will now be defined. It will be referred to as the multi-component model. A state is defined from the values of \( L + H \) state variables describing cows and heifers in the herd. The \( L \) cow variables are defined by the states (in the single-component model) of the \( L \) cows (and empty stalls) of the herd.

Heifers are described by \( H \) state variables defined by age or pregnancy. The values of the heifer state variables are the numbers of heifers in each of the \( H \) states. State 1 represents heifers born at the previous stage, and state \( H \) represents down calving heifers.

Heifers in state 1, \ldots n are young animals before heat detection is initiated, and the state number is equal to the age measured in stages. Heifers in state \( n + 1, \ldots, H - 11 \) are those being submitted for service. Also in this group the state number is equal to the age in stages. Conception is assumed to be independent over stages and animals. Thus, the transition probability \( p_c \) from any of the insemination states (\( n + 1, \ldots, H - 11 \)) to state \( H - 10 \) is assumed to be fixed. The states \( H - 10, \ldots, H \) represent heifers in calf, and a heifer in state \( H - v \) (\( 0 \leq v \leq 10 \)) is expected to calve \( v \) stages ahead.

As it appears, a cow state variable gives the state of one particular cow whereas a heifer state variable gives the number of heifers occupying one particular state. In the work of Ben-Ari and Gal (1986) the cow state variables were defined in the same way as the heifer variables in this study.
That was possible because the single-component model of their study was very small containing only 180 states. Thus, the number of cow state variables was also 180. In the present study the single-component model has got 180000 states and following the number of variables would be the same if the formulation of Ben-Ari and Gal (1986) was used. In the present study, the number of state variables concerning cows is only $L$ (i.e. the maximum number of cows).

An action in the multi-component model is a set of actions $A = (a_1, \ldots, a_L)$ defining the action for each individual cow. The admissible actions are restricted to those where the number of heifers used for replacement does not exceed the number available.

The reward $R_I^A$ of state $I$ under action $A$ is given as

$$R_I^A = \sum_{n=1}^{L} r_{i_n}^a$$

where $i_n$ is the (single-component) state of the $n$th component. In the model, the raising costs of heifers is not included. Instead, calves will be sold to heifers and replacements bought from heifers at market prices.

The transition probability is now considered from state $I$ to state $J$. A state is defined by the cow state variables $c_1, \ldots, c_L$ and the heifer variables $h_1, \ldots, h_H$. Transitions among the heifer states are independent of those among the cow states, so they are treated separately. The cow states are first considered, where the transition probability is just the product of all single-component transition probabilities involved:

$$P_{IJ}^A = p_{i_1,j_1}^A \times p_{i_2,j_2}^A \times \cdots \times p_{i_L,j_L}^A$$

The transition probabilities among the heifer states are determined solely by the combined heat detection and conception rate $p_c$. Thus, it can be seen that the overall transition probabilities exist and might be calculated.

All parameters of the multi-component model have now been defined and it is seen that it is just a usual Markov decision process with known parameters. Only computational capacity prevents the finding of an optimal replacement policy by the usual techniques.

3 THE IMPLEMENTATION OF THE PARAMETER ITERATION METHOD

3.1 Theoretical considerations

In this section an approximative method to circumvent the capacity problems involved in the application of usual techniques is described.
If the actions of individual cows were independent of each others (i.e. at an unlimited supply of heifers) the total present value of the multi-component model \( F_i \) under an optimal policy would equal the sum of all individual present values \( f_{i_k} \), i.e.

\[
F_i = f_{i_1} + \cdots + f_{i_k}
\]

Since they are in fact not independent \( F_i \) is approximated by a function \( G \) involving a number of parameters \( g_1, \ldots, g_n \) relating \( f_{i_1}, \ldots, f_{i_k} \) and \( I \) to \( F_i \). The question is now what kind of function should be preferred? Since the relation is linear in the independent case, Ben-Ari and Gal (1986) argued that it would be natural also in the dependent case to approximate by a linear function. However, since the linearity is caused by the independence assumption this does not seem to be a good choice. Instead, some logical characteristics that the function should possess will be argued for.

1. The total present value under a limited supply of heifers can never exceed the value under unlimited supply as expressed in eqn (3). Thus, \( F_i \) can be expressed as

\[
F_i = f_{i_1} + \cdots + f_{i_k} - G(I)
\]

where \( G \) is a non-negative function.

2. The only way that \( F_i \) may be reduced compared to eqn (3) is by shortage of heifers. Thus, the reduction expressed by \( G \) in eqn (4) must be directly linked to the number of heifers in various states.

3. The reduction caused by shortage of heifers at a certain age (i.e. in a specific state) is decreasing with increasing number of heifers at that age. The reduction caused by a shortage at that age will vanish if the number available is sufficiently large. If no heifers are available at a certain age, the reduction from that age will decrease considerably if one is added. If a second one is added, the reduction will increase further, but not as much as for the first heifer.

4. The size of the reduction caused by a shortage of heifers depends on the composition of the cow herd. If many cows would be replaced in the case of unlimited supply, the reduction will be larger than if only few cows should be replaced. Thus, \( G \) must include some measure of total herd quality. Further, this relation is more prevalent in the case of a shortage of heifers near calving than in the case of a shortage of younger heifers, because the coherence between present and future quality is decreasing with increasing time interval. Thus, the reduction caused by a shortage of new-born heifers will be almost independent of the present cow herd quality, whereas the reduction caused by down calving heifers will be almost entirely linked to the cow herd quality.
(5) Assume that \( n_1 \) heifers of age \( a_1 \) and \( n_2 \) heifers of age \( a_2 \) (where \( a_1 < a_2 \)) are available. The reduction caused by a shortage of heifers at any of the ages depends on the number of heifers as described in point (3), but if \( n_1 = n_2 \), and if the herd quality is assumed to be constant over time, the reduction caused by a shortage of heifers at age \( a_1 \) only differs from the corresponding reduction caused by a shortage of heifers at age \( a_2 \) due to the discount factor. Except for the discounting (the age \( a_2 \) is closer to first calving than \( a_1 \)) the reductions are equal.

3.2 Choice of functions and parameters

One way to express the herd quality \( Q_I \) is to define it from the future profitabilities of the individual cows as defined by Kristensen (1987). It is the benefit (positive or negative) from keeping a cow for at least one additional stage compared to immediate replacement. In other words, a positive future profitability means that the optimal action in the single-component model is to keep, and a negative value means that the optimal action is to replace. The following formula is for (single-component) state \( i \):

\[
q_i^* = r_i^{a_1} + \beta \sum_j p_{ij}^{a_1} f_j - r_i^{a_2} - \beta \sum_j p_{ij}^{a_2} f_j
\]  

(5)

where the superscripts \( a_1 \) and \( a_2 \) refer to the actions keep and replace, respectively. The present value \( f_j \) of a cow in state \( j \) is known from the optimal solution to the single-component problem. It represents the total expected discounted rewards of a Markov decision process starting in state \( j \) and running over an infinite number of stages under an optimal policy. The interpretation of eqn (5) is that from the next stage an optimal policy will be followed, but at the present stage any of the actions may be chosen. The future profitability is then calculated as the difference in present value when the cow is kept for at least one stage compared to the present value of immediate replacement. The symbol \( \beta \) is the discount factor from a stage to the previous one.

A weakness of this definition of quality is that the estimated loss, if the future profitability is negative, is related only to a very short period (one stage). At the next stage it is assumed that a heifer is available, and if the future profitability is still negative, a replacement is assumed to take place. Thus, the future profitability of even the least efficient cow is numerically small (though negative). In other words, a negative future profitability does not indicate whether the cow is just in a temporary crisis or whether it is really not profitable in the long run either.
Instead of eqn (5), an alternative definition of quality shall be considered

\[ q_i = \phi_{it} - r_i^{a_2} - \beta \sum_j p_{ij}^{a_2} f_j \]  

(6)

where \( \phi_{it} \) is the present value (in the single-component model) of a cow which at stage \( t \) is in state \( i \) provided that it is kept at least until next calving. This present value is calculated recurrently according to

\[ \phi_{it} = f_i \]

if a calving takes place in state \( i \), and

\[ \phi_{it} = r_i^{a_1} + \beta \sum_j p_{ij}^{a_1} \phi_{j,t+1} \]

if no calving takes place. The summation at the right-hand side of the equation is over all possible states at stage \( t + 1 \). The superscript \( a_1 \) is the action 'keep' and \( a_2 \) is 'replace'. Thus, \( q_i \) is the advantage (positive or negative) of keeping the cow at least until the next calving. In the calculation of \( \phi_{it} \) the absolute value of \( t \) is of no relevance. The calculation is just started in the states where a calving takes place, and the value for states representing other stages of lactation are then calculated backwards step by step beginning one stage before calving and ending one stage after the previous calving. The advantage of eqn (6) over eqn (5) is that the defined quality refers to a longer period (instead of just one stage). It therefore represents a more permanent characteristic of the animal avoiding a temporary crisis to result in a low-quality classification.

The herd quality in turn is defined as

\[ Q_t = \sum_{n=1}^{L} q_{in} \]  

(7)

In order to get an impression of the shape of the function \( G \), the immediate loss from shortage of down calving heifers (i.e. heifers of state \( H \)) will be considered. From the single-component model we know the future profitability (eqn (5)) of each individual cow in the herd at any time. If no heifers at all are available, the immediate loss will numerically equal the sum of all future profitabilities below zero (i.e. of all cows that would be replaced if the supply of heifers was unlimited). If one and only one heifer is available, the lowest ranking cow will be replaced if its future profitability is negative. The immediate loss thus numerically equals the sum of the remaining future
profitabilities below zero. Correspondingly, the immediate loss if 2, 3, 4 or more heifers are available may be calculated in a similar way. If the number of heifers available exceeds the number of cows having negative future profitabilities the loss will be zero.

By simulation, it is possible to generate a large number of herd combinations and thus a large number of joint observations of total herd qualities and immediate losses if $0, 1, 2, 3, \ldots$ heifers of state $H$ are available. By analysis of such a simulated material it was found that a good fit was obtained by the function

$$g_0(h_H, Q_I) = a \exp(b h_H + c Q_I)$$  \hspace{1cm} (8)$$

where $g_0(h_H, Q_I)$ is the expected immediate loss if the number of heifers just about to calve is $h_H$, and the total herd quality is $Q_I$. The symbols $a$, $b$ and $c$ are parameters to be estimated. It is expected that $a$ will be positive, and $b$ and $c$ will be negative. It appears that eqn (8) becomes linear in $h_H$ and $Q_I$ by using logarithms. Therefore, the values of $a$, $b$ and $c$ may be determined by ordinary least-squares regression.

From these results the expected loss from shortage of heifers in state $H - 1$ (i.e. heifers expected to calve one stage ahead) are now considered. From eqn (8) the expected discounted loss is

$$g_1(h_{H - 1}, Q_I) = \beta E(g_0(h_{H - 1}, Q_I) | I)$$  \hspace{1cm} (9)$$

where the stochastic variable $Q_I$ is the herd quality at the following stage given a present herd quality of $Q_I$. The number of heifers in state $H$ at the following stage equals the number in state $H - 1$ at the present stage. If eqn (8) is substituted into eqn (9), then

$$g_1(h_{H - 1}, Q_I) = \beta E(a \exp(b h_{H - 1} + c Q_I) | I)$$  \hspace{1cm} (10)$$

In a similar way, the expected discounted loss from shortage of heifers in any other state of pregnancy $H - 10, \ldots, H - 2$ may be calculated. The central elements are a discount factor corresponding to the time gap until the heifers of a state are expected to calve and the expected loss at that time given the present herd quality.

The expected loss from heifers not yet pregnant is more complicated to calculate, because the expected number of these heifers to calve $v$ stages ahead is not equal to the number of heifers in a specific state. Young heifers of the same age will typically not conceive at the same time, and heifers of different ages may conceive at the same time. Recalling that states $1, \ldots, n$ represent young heifers before heat detection is initiated, and states $n + 1, \ldots, H - 11$ represent heifers under insemination, it can be concluded that the total number, $H_{11}$, of heifers from the insemination states to calve 11 stages ahead is binomially distributed with the parameters
\[ N = h_{n+1} + \cdots + h_{H-11} \] and \( p = p_c \). Accordingly, the expected value and variance of the total number of heifers to calve 11 stages ahead are
\[
E_{11}(H_{11} \mid I) = p_c \sum_{i=n+1}^{H-11} h_i
\]
\[
V_{11}(H_{11} \mid I) = p_c(1 - p_c) \sum_{i=n+1}^{H-11} h_i
\]

If the total number of heifers to calve 12 stages ahead, \( H_{12} \), is looked at it is found that the number of heifers from state \( n \) is binomially distributed with the parameters \( h_n \) and \( p_c \). The number of heifers to calve 12 stages ahead from the present insemination states is also binomially distributed with the parameters \( h_{n+1} + \cdots + h_{H-12} \) (heifers from state \( H - 11 \) are not included because if they do not conceive at the present stage—calving 11 stages ahead—they are culled) and \( (1 - p_c)p_c \) (i.e. the probability that they conceive at the next stage provided that they do not conceive at the present stage). Thus, the expected value and variance of the total number of heifers to calve 12 stages ahead are
\[
E_{12}(H_{12} \mid I) = h_np_c + (1 - p_c)p_c \sum_{i=n+1}^{H-12} h_i
\]
\[
V_{12}(H_{12} \mid I) = h_n p_c(1 - p_c) + (1 - p_c)p_c(1 - (1 - p_c)p_c) \sum_{i=n+1}^{H-12} h_i
\]

Continuing in the same way it is found that the total number of heifers to calve \( v \) stages ahead \( (v > 10) \) is a sum of binomially distributed random variables having the expected value and variance as follows:
\[
E_v(H_v \mid I) = \sum_{i=n-v+12}^{n'} p_c(1 - p_c)^{(i-n+v-12)}h_i + p_c(1 - p_c)^{v-11} \sum_{i=n+1}^{H-v} h_i
\]
\[
V_v(H_v \mid I) = \sum_{i=n-v+12}^{n'} p_c(1 - p_c)^{(i-n+v-12)}(1 - p_c)(1 - p_c)^{(i-n+v-12)}h_i
\]
\[ + p_c(1 - p_c)^{v-11}(1 - p_c)(1 - p_c)^{v-11} \sum_{i=n+1}^{H-v} h_i \]
where \( n' = \min \{ n, H - v \} \). The limits of the first sum in the equation are from the youngest possible heifers to calve \( v \) stages ahead to either the oldest heifers presently not being observed for heat or the oldest possible heifers to calve \( v \) stages ahead (whichever is lowest). The limits of the second sum are from the youngest heifers presently being observed for heat to the oldest heifers in the insemination states that are not being discarded if they do not become pregnant to calve \( v \) stages ahead (or before). If for a value of \( v \), the lower limit of a sum is higher than the upper limit, the sum will vanish.

The total reduction \( G(I) \) in present value caused by shortage of heifers is calculated as the sum of expected losses from shortage of heifers expected to calve different stages ahead, i.e.

\[
G(I) = \sum_{v=0}^{N'} \beta^v E_v (a \exp (bH_v + cQ,J) | I) \tag{14}
\]

where the random variables \( H_v \) and \( Q,J \) are the total number of heifers calving and the herd quality, respectively, \( v \) stages ahead given the present (multi-component) state \( I \). The correct value of \( N' \) is infinity, but for practical purposes it is reasonable to let \( N' \) be the maximum age (in stages) that a newborn heifer may possibly calve under the defined insemination and culling policy.

In order to calculate \( G(I) \), how to evaluate the expected value at the right-hand side of eqn (14) must be considered. In other words, the distribution of the time series of observed herd qualities at successive stages must be known. It is obvious that if the herd quality is low at the present stage, it must be expected to be low at the following stage too. Further, a large number of heifers to calve at the present stage is assumed to imply a higher herd quality at the next stage, because of the possibilities of replacement. A simple way to model this property is to define the time series as follows, where \( Q,t \) is the herd quality at stage \( t \):

\[
Q_{t+1} = m + fh_H + e_{t+1}
\]

and

\[
e_{t+1} = de_t + e_{t+1} \tag{15}
\]

where \( m \) is the average value under the (multi-component) policy followed if no heifers were available, \( f \) is a parameter describing the marginal improvement caused by an additional heifer, \( d \) is an autoregression coefficient and the residuals \( e_t \) are assumed to be mutually independent and normally distributed with zero mean and a standard deviation of \( \sigma \). The values of \( m,f,d \) and \( \sigma \) will depend on the (multi-component) policy. Thus, at an intensive culling, \( m \) is assumed to be higher (i.e. better herd quality). The value of \( d \) is assumed to be lower because many replacements will decrease
the correlation over stages. Finally, an intensive culling is expected to
decrease the random variation $\sigma$.

The expected value and the variance of the herd quality $v$ stages ahead,
given the present state $I$, are calculated as follows:

$$
E_v(Q_J | I) = m + fE_v(H_v | I) + d^v(Q_I - m - fh_H) \\
V_v(Q_J | I) = f^2 V_v(H_v | I) + \sigma^2 \frac{1 - d^{2v}}{1 - d^2}
$$

(16)

where the conditional expectation and variance at the right hand sides are
known from eqn (13).

Given the model in eqn (11), an approximate value of the right-hand side of
eqn (9) can be calculated. For a given (multi-component) state $I$, the
expression $bH_{v+1} + cQ_J$ is normally distributed with an expected value of $bh_{H-1} + 
(c(m + fh_{H-1} + d(Q_I - m - fh_H))$ and a variance of $c^2\sigma^2$. Consequently, the
distribution of $\exp(bh_{H-1} + cQ_J)$ is log-normal with an expected value of
$\exp(bh_{H-1} + c(m + fh_{H-1} + d(Q_I - m - fh_H)) + c^2\sigma^2/2)$. For all values of
$v < 11$ the expression $bH_v + cQ_J$ is normally distributed under the
assumptions used) and therefore the distribution of the exponential value is
log-normal) with the expectation and variance given as

$$
E_v(bH_v + cQ_J | I) = bh_{H-v} + c(m + fh_{H-v} + d^v(Q_I - m - fh_H)) \\
V_v(bH_v + cQ_J | I) = c^2\sigma^2 \frac{1 - d^{2v}}{1 - d^2}
$$

(17)

For values of $v \geq 11$ the situation is more complicated since $H_v$ in this
situation is a random variable, which is a sum of several binomially
distributed random variables. Therefore, the expression $bH_v + cQ_J$ is not
normally distributed. Since, however, the normal distribution is usually a
good approximation of a binomial distribution, and further $H_v$ is a sum of
several random variables, the expression is assumed to be approximately
normally distributed. The mean and variance may be calculated as

$$
E_v(bH_v + cQ_J | I) = bE_v(H_v | I) + cE_v(Q_J | I) \\
V_v(bH_v + cQ_J | I) = b^2V_v(H_v | I) + c^2V_v(Q_J | I) + 2bcV_v(H_v | I)
$$

(18)

where the conditional means and variances of the right-hand side are known
from eqns (13) and (16). Accordingly

$$
E_v(\exp(bH_v + cQ_J | I) = \exp(E_v(bH_v + cQ_J | I) + \frac{1}{2}V_v(bH_v + cQ_J | I))
$$

If this equation is used in eqn (14) the following is obtained:

$$
G(I) = a \sum_{v=0}^{N'} \beta^v \exp(E_v(bH_v + cQ_J | I) + \frac{1}{2}V_v(bH_v + cQ_J | I))
$$

(19)
The (multi-component) future profitability of a cow is

\[ \pi_t = q_t^* + G(I^1) - G(I^2) \]  

(20)

where \(I^1\) and \(I^2\) are the multi-component states if the cow is kept or replaced, respectively, and \(q_t^*\) is the future profitability defined in eqn (5) for the single-component model. An approximately optimal policy of the multi-component model is given by the optimal policy of the single-component model combined with the parameters of the function \(G\). A cow is replaced if the future profitability of eqn (14) is negative and kept otherwise. Each time the lowest ranking cow has been replaced, the future profitabilities of the remaining cows must be recalculated under the new herd quality and the reduced number of heifers caused by the replacement.

3.3 An approximate solution

Having chosen the functions and parameters, the steps involved in the determination of an approximately optimal policy of the multi-component model may be described:

1. Calculate an optimal policy of the single-component model by usual methods.
2. Estimate the parameters \(a, b\) and \(c\) of eqn (7) from a simulated data set using an arbitrary policy. The parameters \(m, d, f\) and \(\sigma\) of eqn (11) are also estimated.
3. Simulate a time period using the present parameters \(m, d, f\) and \(\sigma\) to define the policy to be followed. Calculate the economic result of the simulated period and estimate new values of \(m, d, f\) and \(\sigma\) from the simulated data set.
4. Repeat step (3) until the results have stabilised.

3.4 Culling of heifers

In the previous sections, the problem of optimal replacement of cows, when a given number of heifers in different states are available, has been considered. The opposite problem of whether a heifer in a specific state should be sold or raised for future milk production in the herd is now considered. In that connection, the production value \(v(i, I)\) of a heifer in state \(i\) given the multi-component state \(I\) is defined as

\[ v(i, I) = G(I_1) - G(I) \]  

(21)

where \(I_1\) is the multi-component state if one heifer in state \(i\) is sold. The value of \(G(I)\) is calculated from eqn (19). The production value expresses the expected future contribution of the heifer to net returns from the production of cows. This value is calculated from the point of view of the milk producer.
Further on, the alternative value is defined as the present market value of the heifer plus the expected discounted costs of raising the heifer from present age to first calving minus the discounted price of a down calving heifer. The alternative value expresses the gain (positive or negative) from selling the heifer immediately compared to keeping it until just before calving and then sell it or include it in the cow herd (which ever is best at that time). The alternative value is calculated from the point of view of the heifer producer.

However, the whole idea of the multi-component model is that the heifer producer and the milk producer is one and the same. On considering the sale of a heifer, the production value should be compared to the alternative value. If the alternative value is higher it is profitable to sell the heifer; otherwise, it should be kept for future replacement.

4 TEST OF THE METHOD

4.1 Material and methods

In order to test the method described in the previous section a single-component dairy cow replacement model and a stochastic simulation model of a dairy herd is needed. As mentioned in section 2, the replacement model used is the one described by Kristensen (1989). The simulation model is the one used by Kristensen and Thysen (1991). It includes, for each cow, the same traits as the replacement model. Further, it includes the heifers of the herd. The main characteristics of the simulation model are shown in Table 1.

A standard set of prices and herd level of milk yield was defined as in Table 2. In order to test the method under various conditions, 100 sets of alternatives have been generated by a random number generator. In each set of conditions the individual prices and level of milk yield were drawn independently from uniform distributions over intervals defined from the original values of Table 2 ± 15%.

Under each of the 100 sets of conditions the method of section 3.3 was applied. In the simulation of step (2), the cows were ranked according to their single-component future profitability as defined in eqn (5). Thus, a cow is replaced if the future profitability is negative and a heifer is available. In the simulation of step (3), the cows were ranked according to their multi-component future profitability as defined in eqn (20). (The ranking was recalculated each time a replacement was performed). In all cases a herd of 100 cows (as a maximum) is simulated over a period of 100 years in order to decrease the random variation on results.

Step (1) of the method gives the economic result if an optimal policy is
### TABLE 1
Main Characteristics of the Simulation Model

<table>
<thead>
<tr>
<th>Cows</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of 4-week stages per lactation</td>
<td>11–18</td>
</tr>
<tr>
<td>Maximum number of lactations per cow</td>
<td>6</td>
</tr>
<tr>
<td>Forced replacement if not pregnant before</td>
<td>238 days</td>
</tr>
<tr>
<td>Stochastic state variables</td>
<td></td>
</tr>
<tr>
<td>Breeding value (milk yield) of father</td>
<td>5 classes</td>
</tr>
<tr>
<td>Milk yield, previous lactation</td>
<td>15 classes</td>
</tr>
<tr>
<td>Milk yield, present lactation</td>
<td>15 classes</td>
</tr>
<tr>
<td>Length of calving interval</td>
<td>8 classes</td>
</tr>
<tr>
<td>Deterministic state variables</td>
<td></td>
</tr>
<tr>
<td>Lactation stage</td>
<td>18 classes</td>
</tr>
<tr>
<td>Lactation number</td>
<td>6 classes</td>
</tr>
<tr>
<td>Total number of states (approximately)</td>
<td>180,000</td>
</tr>
<tr>
<td>Probability of a calving to result in a surviving heifer</td>
<td>0.45</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Heifers</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>State variable</td>
<td></td>
</tr>
<tr>
<td>Age or reproductive status</td>
<td>41 classes</td>
</tr>
<tr>
<td>Minimum age of first breeding</td>
<td>56 weeks</td>
</tr>
<tr>
<td>Probability of conception per stage</td>
<td>0.33</td>
</tr>
<tr>
<td>Age of disposal of open heifers</td>
<td>116 weeks</td>
</tr>
</tbody>
</table>

*From Kristensen and Thysen (1991).*

### TABLE 2
Standard Prices and Herd Level of Milk Yield

<table>
<thead>
<tr>
<th>Prices (Dkr)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Milk (kg FCM&lt;sup&gt;a&lt;/sup&gt;)</td>
<td>2.40</td>
</tr>
<tr>
<td>Basic feed (SFU&lt;sup&gt;b&lt;/sup&gt;)</td>
<td>1.30</td>
</tr>
<tr>
<td>Feed for milk production (SFU)</td>
<td>1.45</td>
</tr>
<tr>
<td>Calf</td>
<td>1400.00</td>
</tr>
<tr>
<td>Heifer</td>
<td>9000.00</td>
</tr>
<tr>
<td>Young cow (kg live weight)</td>
<td>11.50</td>
</tr>
<tr>
<td>Older cow (kg live weight)</td>
<td>11.00</td>
</tr>
<tr>
<td>Interest rate (corrected for tax and inflation, %)</td>
<td>3.00</td>
</tr>
<tr>
<td>Herd level of milk yield (week 1–40, 1st lactation)</td>
<td>5800.00</td>
</tr>
</tbody>
</table>

<sup>a</sup> Fat corrected milk.

<sup>b</sup> Scandinavian feed unit.
followed and an unlimited supply of heifers is available. This result thus
gives an upper bound of the economic result using the new method. A lower
bound is given by the economic result of the simulation of step (2), where the
single-component future profitability is used as replacement criterion under
limited supply of heifers. The new method should do better than that in
order to be relevant.

The standard conditions are used for calculating the production values of
heifers in herds of varying composition.

4.2 Results

In all 100 sets of conditions step (3) of the method was run three times in
order to see when the results stabilised. The economic results, however, did
not improve by running step (3) more than once. On the other hand, the
values of the parameters \(m, d, f\) and \(\sigma\) often (but not always) changed from
step (2) to step (3), but only slightly from first to second run of step (3). All
results in the following are taken from the second run of step (3). In Table 3
the parameter estimates under standard conditions are shown.

Denote as \(O_i\) \((i = 1, \ldots, 100)\) the economic result in Danish kroner (Dkr)
per cow per year (revenues from milk, calves and culled cows minus the costs
of feeds and heifers) under an optimal policy for the \(i\)th set of conditions
assuming unlimited supply of heifers. These results are the expected values
calculated directly from the functional equations of the single-component
Markov decision process. The simulation results for the \(i\)th set of conditions
under limited supply of heifers are denoted as \(S_i\) and \(M_i\) using the single- and
multi-component future profitabilities, respectively. Unlike \(O_i, S_i\) and \(M_i\) are

| TABLE 3 |
| Parameter Estimates of the Models in Eqns (8) and (15) under the Standard Conditions of Table 2 |

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Step 2</th>
<th>Step 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>First run</td>
<td>Second run</td>
</tr>
<tr>
<td>Level</td>
<td>(a)</td>
<td>(3 \times 10^7)</td>
<td></td>
</tr>
<tr>
<td>Effect of heifers</td>
<td>(b)</td>
<td>(-0.544)</td>
<td></td>
</tr>
<tr>
<td>Effect of herd quality</td>
<td>(c)</td>
<td>(-0.670 \times 10^{-4})</td>
<td></td>
</tr>
<tr>
<td>(R^2) of eqn (8)</td>
<td></td>
<td>(0.71)</td>
<td></td>
</tr>
<tr>
<td>Basic level</td>
<td>(m)</td>
<td>(1.43 \times 10^5)</td>
<td>(1.45 \times 10^7)</td>
</tr>
<tr>
<td>Autoregression coefficient</td>
<td>(d)</td>
<td>(0.762)</td>
<td>(0.784)</td>
</tr>
<tr>
<td>Effect of heifers</td>
<td>(f)</td>
<td>(1675)</td>
<td>(1946)</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>(\sigma)</td>
<td>(1.17 \times 10^4)</td>
<td>(1.18 \times 10^4)</td>
</tr>
<tr>
<td>(R^2) of eqn (15)</td>
<td></td>
<td>(0.64)</td>
<td>(0.68)</td>
</tr>
</tbody>
</table>
not expected values, but only estimates of the true expectations. Therefore, a
certain (limited) variation around the true values is unavoidable. In Table 4
the percentiles, extremes and means over the 100 sets of conditions are
shown for $O_1, \ldots, O_{100}$ as well as for the differences $S_1 - O_1, \ldots, S_{100} - O_{100}$
and $M_1 - O_1, \ldots, M_{100} - O_{100}$. Finally, the same statistics are shown for the
differences $M_1 - S_1, \ldots, M_{100} - S_{100}$ representing the benefits of the multi-
component model over the single-component model.

As it appears, the results from the multi-component model are on average
60 Dkr better per cow per year than when the single-component future
profitabilities are used as replacement criterion. The results using the single-
component criterion are on average 200 Dkr below the unrestricted optimal
solution and those using the multi-component criterion are on average 140
Dkr below the unrestricted optimal solution per cow per year.

As it appears from Table 4, the benefit of the multi-component
method over the single-component varies considerably over the 100 sets of
conditions. It should be expected that the benefit is highest in situations
where the supply of heifers is smaller than the optimal need for
replacements. The most important factor determining the level of
replacement is the price difference between a heifer and the carcass value of a
cow (Kristensen & Østergaard, 1982). Almost equivalent to this difference is
the replacement costs defined as the price of a heifer minus the slaughter
value of a young cow and the value of a calf.

In Fig. 1, the benefit of the method is plotted against the replacement costs
under each of the 100 sets of conditions. As expected, the benefit is very high
in situations with low replacement costs, where the optimal level of
replacement is high, and the heifers available do not suffice. In situations
with low replacement costs the benefit seems almost to vanish. In some cases
Fig. 1. Plot of the benefit (Dkr) of the multi-component model over the single-component model against the replacement costs (Dkr—see the text for definition).

the benefit is even negative, but, since the values represent the results of stochastic simulation, at least some negative values should be expected when the true value is numerically small though non-negative. By repetitive simulation runs under fixed conditions the standard deviation of a simulation result was determined to be around 30 Dkr. Since the benefits in Fig. 1 represent differences between two results the standard deviation in the direction of the y-axis becomes 42 Dkr. The negative values are therefore easily explained as results of random variation around a non-negative true value.

In order to study the production values of heifers in relation to present herd quality and heifer state (age or pregnancy status) a heifer stock was designed in a manner that rather precisely results in the same expected
number of heifers to calve at each future stage 40 stages ahead. Under the standard conditions the herd quality was successively varied from the average value minus 2 units of standard deviation in steps of 0.5 units to the average value plus 2 units. For each heifer state the production value of the last heifer was determined as the loss in expected discounted future net returns from cows if exactly one heifer from the state was culled. The parameter estimates of Table 3 (step (3), first run) were applied.

In Fig. 2 the production value is plotted against heifer state and herd quality. As expected the production value depends heavily on the herd quality for heifers which are soon going to calve. For young heifers the effect vanishes.

In Fig. 3 the production value of the last heifer in a state is plotted against the number of heifers in the state and the state number. The basis is the same heifer composition as used in Fig. 2, but the number of heifers has been varied from 1 to 9 one state at a time. The calculations have been performed for 2 levels of present herd quality resulting in two different plots.

**Fig. 2.** The production value (P) of a heifer in Dkr as a function of present herd quality (unit: standard deviation, 0 = mean value) and heifer state (age or pregnancy status). State 1 represents newly born heifers and state 40 represents heifers just about to calve. The basic heifer stock is constructed so that the supply of heifers per stage is constant.
Fig. 3. Plot of production value ($P$) in Dkr of the last heifer in a state as a function of the number of heifers in the state and the state number. States 1–13 represent young heifers before heat detection is initiated, states 14–28 represent heifers under insemination and states 29–38 represent pregnant heifers.
In both cases the plot is divided into three sections, which are states 1–13 representing young heifers not yet observed for heat, states 14–28 representing heifers under insemination, and states 30–39 representing heifers in calf. In the first two sections the effect of the number of heifers in a state is much smaller than for pregnant heifers. The reason is that for these young animals, the heifers of a particular state are not expected to calve at the same time because of the random variation in heat detection and conceiving. Thus, few heifers in one state is to a large extent compensated by sufficient heifers in other states. A similar compensation is not possible for heifers in calf and therefore shortage of heifers in a state is far more critical in those cases.

The reason for the lower production value of heifers under insemination compared to young heifers before insemination is that, for instance, five heifers in an insemination state are relatively more than the same number in one of the states from 1 to 13. The expected number of heifers in the youngest states is the average number of heifers born at a particular stage (4-week period). The expected number of heifers in an insemination stage is lower because some of the heifers at the age in question have already conceived and thus are transferred to a pregnancy state.

At high herd quality, the future quality is expected to be lower than it is at present, and therefore the production value of young heifers in calf is higher than for down calving heifers. At low quality the situation is opposite as it appears from Fig. 3.

5 DISCUSSION

This study is a contribution to the practical solution of the multi-component replacement problem in dairy cattle, where the limited supply of heifers as mentioned by Ben-Ari et al. (1983) complicates the problem compared, for example, to replacement of industrial items. An approximate method is suggested, since the calculation of an exact solution is prohibitive. Even the approximate method is very time consuming on the computer. The main reasons are the size of the single-component model used and the simulations of steps (2) and (3). If a single-component model of that size is used, we must conclude that at present the calculations are too comprehensive for direct practical application on a dairy herd. On the other hand, the calculations were performed on a (powerful) PC, which in a few years probably is a standard equipment on a commercial dairy farm. When further multi-tasking computer systems come into general use, the time spent on a single job is not so important because the computer may be used simultaneously for other purposes. Therefore, a method as the one described may very well be applied in a future decision support system concerning replacement in dairy herds.
this connection it should be noticed, that as long as the price conditions are the same, the optimal policy is represented solely by the optimal solution to the unrestricted single-component problem combined with the parameter estimates concerning \( a, b, c, m, d, f \) and \( \sigma \) of the function \( G(l) \) in eqn (20). Therefore, a new optimal policy of the multi-component problem only has to be calculated if prices change.

It is not possible to compare the results to those derived from exact solutions to the multi-component model. However, the alternative to the multi-component model is to use the future profitabilities from the single-component model as replacement and ranking criterion, and we are able to compare the results to this alternative. From Fig. 1 and Table 4 it can be concluded, that for low replacement costs (i.e. when the need for heifers exceeds the supply) the multi-component model improves the economic result considerably compared to the usual single-component model. For high replacement costs (where the supply of heifers is sufficient) the result is almost the same no matter if the single or multi-component model is applied. The reason is that for increasing number of heifers the multi-component future profitability of eqn (20) converges towards the value of the single-component future profitability of eqn (5) thus making the two criteria equivalent if the number of heifers is sufficiently large.

In such a situation the only advantage of the multi-component model is the information concerning culling of heifers as described in section 3.4. The information is given as the production value of the heifer, and the major force of the method is that it places the heifer in the herd environment where it belongs. In a herd where heifers are bought at the market, the value of a heifer is just the market price which only depends on the state of the animal in question. On the other hand, if only home-grown heifers are used as replacements (for the reasons mentioned in section 1) the value of a heifer can not be determined by calculations only relating to that particular animal. Instead, the following questions will have to be considered: How many other heifers at similar age have we got? What is the future need for replacements (expressed by the current herd quality)? The value of the heifer heavily depends on the answers to these questions, and this dependency is directly taken into account in the multi-component model.

An examination of the effects in Figs 2 and 3 confirms that the production value of pregnant heifers depends very much on the present herd quality and the number of heifers in a state. For younger heifers the effect of present herd quality almost vanishes because of the decreasing correlation between present and future quality over increasing time lag. The effect on number of heifers in the particular state in question is still present but to a much smaller extent than for pregnant heifers. The production value of the heifers may be calculated in any situation, and in combination with the alternative value as
defined in section 3.4, it is very important in the decisions concerning how many heifers to raise for future replacement. Also, in a situation with limited housing capacity the information is relevant. Since the value of this information concerning heifers is even more important in a situation with high replacement costs (because the supply of heifers in those cases exceeds the demand), it is relevant to use the multi-component model even in a situation with high replacement costs, where the benefit in the cow herd is small.

As concerns the goodness of fit of the approximations in eqns (8) and (15), the regression analyses resulted in $R^2$ values of 0.71 in eqn (8) and in eqn (15) the values varied from 0.64 to 0.68 under the standard conditions (Table 3). The estimates of the autoregression coefficient of eqn (15) varied from 0.76 to 0.79 (standard conditions, Table 3) showing a high degree of autocorrelation in herd quality over time. All effects in the models were highly significant, and the parameters were very precisely estimated. Thus, the overall impression is that the models used in the approximations seem to fit quite well.

The supply of heifers has been identified as a limiting restraint on the replacement problem in many dairy herds. The basic idea of the multi-component approach is to consider in what way the limiting restraint logically affects the known optimal solution to the unrestricted problem. Then the influence of the restraint is approximated by a function $G(I)$ having the desired logical properties, and finally the parameters of the function are estimated from simulated data.

There are several other limiting restraints on the replacement problem in dairy herds. The most obvious one at the time being is the milk quota, but also feed supply and/or labour might be considered. Similar problems exist in other multi-component systems as for instance herds of other animal species. A much wider range of problems involving replacement combined with general resource allocation is then faced, and it is relevant to consider whether the multi-component approach of this paper also might be used in a solution of such problems. It seems natural to expect that the basic idea of this study is applicable in any multi-component replacement problem subjected to some limiting restraint, but the actual choice of the function $G(I)$ depends on the specific problem. Thus, the kind of function used in this study may not apply to other problems.

REFERENCES


