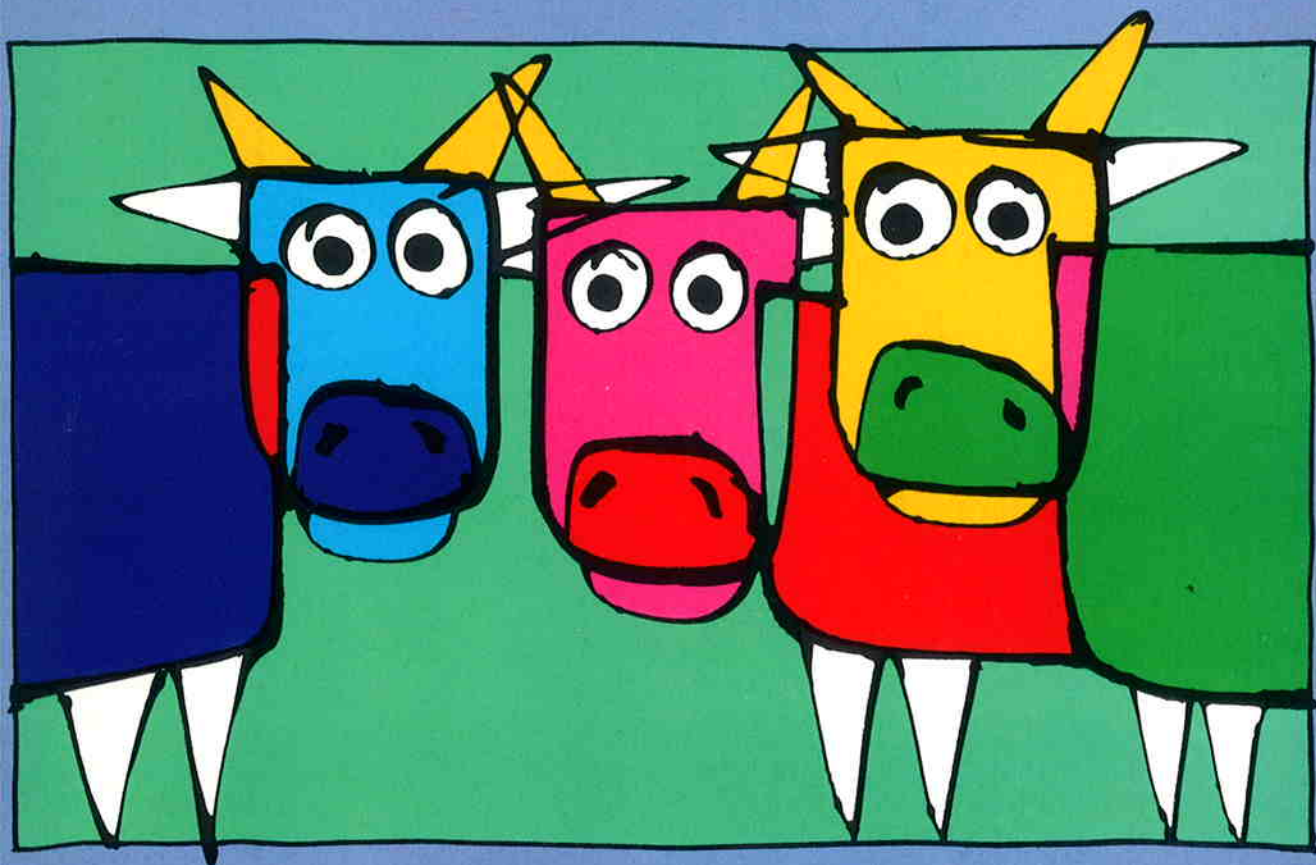


# Markov decision programming techniques applied to the animal replacement problem

*Anvendelse af teknikker for Markov beslutnings-programmering  
til løsning af udskiftningsproblemet vedrørende husdyr*



**Anders Ringgaard Kristensen**

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Anders Ringgaard Kristensen

Dina KVL  
Department of Animal Science and Animal Health  
The Royal Veterinary and Agricultural University  
Copenhagen 1993

Denne afhandling er af Den Kgl. Veterinær- og Landbohøjskoles husdyrbrugs- og veterinærvidenskabelige kollegium antaget til offentligt at forsvares for den jordbrugsvidenskabelig doktorgrad.

København, den 4. maj 1993

*Poul Hyttel*

Formand for det husdyrbrugs- og veterinærvidenskabelige kollegium

## **Til min familie**

### **Markov decision programming techniques applied to the animal replacement problem**

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# Preface

The research behind this thesis was carried out at the Royal Veterinary and Agricultural University partly while I was the holder of a senior research fellowship at the Department of Mathematics from 1985 to 1986 and partly during my employment at the Department of Animal Science and Animal Health, where I was assistant professor from 1986 to 1990 and now hold a position as associate professor. I am very grateful to the staff of both departments for excellent working conditions.

Professor Mats Rudemo, D. techn. Sc., and Professor Mogens Flensted-Jensen, D. Sc., have been very good advisers as concerns the difficult job of writing scientific papers and their scientific support has been a continuous encouragement. I am grateful to the head of my present department, Associate Professor Poul Henning Petersen, Ph. D. (agric.), for his awareness of the importance of management and informatics in animal science. I am indebted to my colleagues, Associate Professor Poul Jensen, M. Sc., and Jens Noesgaard Jørgensen, Ph. D. (agric.), for mutual exchange of computer power and for many animating discussions concerning computers and statistical methods. Associate Professor Sven Bresson, Ph. D. (agric.), and Professor A. Neiman-Sørensen, D.V.M., have indirectly been a great help through their concise ideas concerning the methods and nature of research.

Also my former place of work, the National Institute of Animal Science, Section of "Multidisciplinary Studies in Cattle Production Systems" (*Helårsforsøg med Kvæg*), has been a continuous inspiration. The enthusiasm and scientific competence, which over a short period has raised the section from the "demonstration farm" level to a leading position in Europe concerning the research in cattle production systems and management, has been an ever lasting example. In particular, thanks are due to the head of the section, Vagn Østergaard, D. Sc. (agric.), Iver Thysen, Ph. D. (agric.), Jan Tind Sørensen, Ph. D. (agric.), and Jens Peter Hansen, M. Sc. (agric.) for many inspiring discussions and for comments on earlier versions of several of the chapters of this thesis. Erik Jørgensen, Ph. D. (agric.) at the Department of Research in Pigs and Horses of the same institute has also supplied inspiring suggestions.

As concerns the choice of subject of the thesis, I am indebted to Professor Harald B. Giæver, Agricultural University of Norway, who must suffer the indignity of being referred to as "Giaeever" in this and other studies. His thesis on "Optimal dairy cow replacement policies" from Berkeley arouse my interest in the animal replacement problem already when I was a student. Even though the thesis was published already in 1966, it remains even today an important reference, and several more recent studies have not even reached its level.

The works of Dr. Yaron Ben-Ari from Israel have been the direct inspiration of one of the chapters of this thesis, and indirectly they have inspired several chapters. Also the numerous works of the Department of Farm Management, Wageningen Agricultural University, have been of great value to my research. In particular I am indebted to the works of Professor, dr. ir. Aalt A. Dijkhuizen and his staff.

For typing of some of the manuscripts I thank Mrs. Ruth Crifling, Mrs. Kirsten Astrup, Mrs. Britta Christensen, and for giving advice concerning the English language I thank Mrs. Lone Høst, Mrs. Alice Jensen and Mr. Bent Grønlund.

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Copenhagen, September 1992  
Anders Ringgaard Kristensen



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# Introduction

In the late fifties Bellman (1957) published a book entitled “*Dynamic Programming*”. In the book he presented the theory of a new numerical method for the solution of *sequential decision problems*. The basic elements of the method are the “*Bellman principle of optimality*” and *functional equations*. The idea may be illustrated as follows.

Consider a system being observed over a finite or infinite time horizon split up into periods or *stages*. At each stage, the *state* of the system is observed, and a *decision* (or an *action*) concerning the system has to be made. The decision influences (deterministically or stochastically) the state to be observed at the next stage, and depending on the state and the decision made, an immediate *reward* is gained. The expected total rewards from the present stage until the end of the planning horizon is expressed by a *value function*. The relation between the value function at the present stage and the one at the following stage is expressed by the *functional equation*. Optimal decisions depending on stage and state are determined backwards step by step as those maximizing the right hand side of the functional equation. This way of determining an optimal *policy* is based on the Bellman principle of optimality which says: “An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision” (Bellman, 1957 p. 83).

During the following years Bellman published several books on the subject (Bellman, 1961; Bellman and Dreyfus, 1962; Bellman and Kalaba, 1965). The books were very enthusiastic, and the method was expected to be the solution to a very wide range of decision problems of the real world. The expectations were so great, and they were adduced with such a conviction, that Johnston (1965) ironically compared dynamic programming to a new religion. Others regarded the method as a rather trivial computational device.

Similar stories might be told regarding other new numerical methods, as for instance linear pro-

gramming. However, after some years, the applicational scopes of the methods are encircled. Most often the conclusion is that the method is neither an all-embracing technique nor a triviality. Between these extremities a rather narrow range of problems remains where it is a powerful tool. Other problems are, in some cases, not suitable to be solved by the method. In other cases alternative methods are better.

This also turned out to be the case in dynamic programming. One of the basic elements of dynamic programming is the sequential approach, which means that it fits sequential decision problems best. An obvious example of a sequential decision problem is *the replacement problem*. If an asset is used in a production process it is relevant to consider at regular time intervals whether the present asset should be replaced or it should be kept for an additional period. Thus dynamic programming is a relevant tool, but if the traits of the asset are well defined and their precise behavior over time is known in advance, there are other methods that might be applied to determine the optimal replacement time analytically. On the other hand, if the traits of the asset are affected by *random* variation over time and among assets (as it is the case when the asset is an animal), the replacement decision will depend on the present observations of the traits. In that case dynamic programming is an obvious technique to be used in the determination of an optimal replacement policy.

Having identified dynamic programming as a relevant method to be used with the animal replacement problem, we shall continue on the historical development. In 1960 Howard published a book on “*Dynamic Programming and Markov Processes*”. As will appear from the title, the idea of the book was to combine the dynamic programming technique with the mathematically well established notion of a *Markov chain*. A natural consequence of the combination was to use the term *Markov decision process* to describe the notion. Howard (1960) also contributed to the solution of infinite stage problems, where the *policy itera-*

*tion method* was created as an alternative to the stepwise backward contraction method, which Howard called *value iteration*. The policy iteration was a result of the application of the Markov chain environment and it was an important contribution to the development of optimization techniques.

The policy iteration technique was developed for two criteria of optimality, namely maximization of total expected *discounted* rewards and maximization of expected *average* rewards per stage. Later on, Jewell (1963) presented a policy iteration technique for the maximization of average rewards over time for *semi-Markov decision processes*, which are Markov decision processes of which the stage length is a random variable. Howard (1971) presented a value iteration method for semi-Markov decision processes.

For the sake of completeness it should also be mentioned that *linear programming* was early identified as an optimization technique to be applied to Markov decision processes as described by, for instance, Hadley (1964), but no animal replacement models known to the author have applied that technique. This is in accordance with a conclusion of White and White (1989) that policy iteration (except in special cases) is more efficient than linear programming.

Since the publication of the first mentioned book by Howard (1960) an intensive research in Markov decision programming has been carried out. Many results have been achieved concerning the relations between the various optimization techniques and criteria of optimality. Reviews of these developments are given by van der Wal and Wessels (1985) as well as White and White (1989).

Already three years after the book by Howard (1960), an application to the dairy cow replacement problem was published by Jenkins and Halter (1963). Their model only included the trait "lactation number" (at 12 levels), and the permanent value of the study was only to illustrate that Markov decision programming is a possible tool to be applied to the problem. A few years later, however, Giaever (1966) published a study which represents a turning-point in the application of the method to the animal (dairy cow) replacement problem. He considered all three optimization techniques (value iteration, policy iteration and linear programming), described how to ensure that all mathematical conditions were satisfied, and presented an eminent model to describe the production and feed intake

of a dairy cow. The work of Giaever (1966) has not got the credit in literature that it deserves (maybe because it is only available on microfilm). In a review by van Arendonk (1984) it is not even mentioned.

During the following 20 years, several dairy cow replacement models using Markov decision programming were published, but from a *methodological* point of view none of them have contributed anything new compared to Giaever (1966). Several studies, however, have contributed in *other* ways. Smith (1971) showed that the rather small model of Giaever (1966) with 106 states did not represent the upper limit. His state space included more than 15 000 states. Kristensen and Østergaard (1982) as well as van Arendonk (1985; 1986) and van Arendonk and Dijkhuizen (1985) studied the influence of prices and other conditions on the optimal replacement policy. Other studies (Killen and Kearney, 1978; Reenberg, 1979) hardly reached the level of Jenkins and Halter (1963). Even though the sow replacement problem is almost identical to that of dairy cows, very few studies on sows have been published. The only exceptions known to the author are Huirne (1990) and Jørgensen (1992).

A study of Ben-Ari et al. (1983) deserves special attention. As regards methodology it is not remarkable, but in that study the main difficulties of the animal replacement problem were identified and clearly formulated. Three features were mentioned:

- 1) *Uniformity*. The traits of an animal are difficult to define and measure. Furthermore the random variation of each trait is relatively large.
- 2) *Reproductive cycle*. The production of an animal is cyclic. It has to be decided *in which* cycle to replace as well as *when* to replace inside a cycle.
- 3) *Availability*. Only a limited supply of replacements (heifers or gilts) is available.

The first feature in fact covers two different aspects, namely *uniformity* because the traits are difficult to define and measure, and *variability* because the traits vary at random among animals and over time. The third feature is an example of a *herd restraint*, i.e. a restriction that applies to the herd as a whole and not to the individual animal. Other examples of herd restraints are a production quota or

a limited housing capacity. We shall therefore consider the more general problem of herd restraints.

We may conclude that when the research presented in this thesis was initiated, the methodological level concerning the application of Markov decision programming to the animal replacement problem was represented by Giaever (1966). The main difficulties that the method should overcome had been identified by Ben-Ari et al. (1983). If we compare the approach of Giaever (1966) to the difficulties that it ought to solve, we may conclude that the problems related to *variability* are directly solved, and as it has been shown by Kristensen and Østergaard (1982) as well as van Arendonk (1985), the problems concerning the *cyclic production* may readily be solved without any methodological considerations. The only problem concerning variability and cyclic production is that to cover the variability the state variables (traits) have to be represented by many levels, and to deal with the cyclic production a state variable representing the stage of the cycle has to be included. Both aspects contributes significantly to an explosive growth of the state space. We therefore face a *dimensionality* problem. Even though all necessary conditions of a Markov decision process are met, the solution in practice is prohibitive even on modern computers. The problems concerning uniformity and herd restraints are *not* solved by the approach of Giaever (1966).

The purpose of this thesis is to adapt the Markov decision programming techniques to be able to cope with the animal replacement problem in a satisfactory way. The problems to be solved (totally or partially) have been identified as the dimensionality problem, the uniformity problem and the problems caused by herd restraints. A secondary purpose is to illustrate and discuss the applicational perspectives of the techniques. All numerical results of the thesis refer to dairy cows, but recently Markov decision programming has also been applied to sows (Huirne, 1990; Jørgensen, 1992). Since the sow replacement problem does not differ very much from that of dairy cows, the same methodological problems arise, and the results of this thesis are therefore relevant in sow replacement models too.

In Chapter II a systematic survey of the developed techniques is given. In Chapters III-VIII the individual techniques are described in details. The applicational perspectives are discussed in Chapter

IX, and Chapters X and XI are examples of such applications.

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# A survey of Markov decision programming techniques applied to the animal replacement problem<sup>1</sup>

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## Abstract

The major difficulties of the animal replacement problem are identified as uniformity, herd restraints and the “curse of dimensionality”. Approaches for circumventing these difficulties using Markov decision programming methods are systematically discussed, and possible optimization techniques are described and evaluated. Assuming that the objective of the farmer is maximum net returns from the entire herd, relevant criteria of optimality are discussed. It is concluded that a Bayesian technique is a promising approach as concerns the uniformity problem, that parameter iteration may be used under herd restraints, and that hierarchic Markov processes has contributed to the solution of the dimensionality problem.

**Keywords:** Criteria of optimality, hierarchic Markov process, parameter iteration, Bayesian updating.

## 1. Introduction

This paper deals with a *problem* and a *technique*. The problem is the determination of optimal *replacement of animals* (in practice limited to cows and sows). The technique is *dynamic programming* or, to be more specific, *Markov decision programming*. The literature on the replacement problem in *general* is very extensive. Studies on the *animal replacement problem* are also numerous, but naturally they are fewer than for the general problem. A review of studies on dairy cow replacement is given by van Arendonk (1984). Also on Markov decision programming the literature is extensive. Recent reviews are given by van der Wal and Wesels (1985) as well as White and White (1989). A review of applications to agriculture has been given by Kennedy (1981).

Since *both* the problem and the technique discussed in this paper seem to be well elucidated in the literature, a relevant question to ask would be why the *combination* of the problem and the tech-

nique should be the subject of a survey. The answer is that animal replacement problems differ from general replacement problems in several respects, and in order to deal with the problems arising from this observation many modifications of the general Markov decision programming technique are relevant or even necessary.

The general replacement theory most often implicitly assumes industrial items as the objects of replacement. Ben-Ari et al. (1983) mention three main features in which the dairy cow replacement problem differs from the industrial problem. Exactly the same features are relevant in sow replacement models.

- *Uniformity*. It is a problem that the traits of an animal are difficult to define and measure. Furthermore the variance of each trait is relatively large.
- *Reproductive cycle*. The production of an animal is cyclic. We therefore need to decide in *which* cycle to replace as well as when to replace *inside* a cycle.
- *Availability*. Usually there is a limited supply of replacements (heifers or gilts). This is particularly the case when the farmer only uses home-

<sup>1</sup> This research was carried out as part of Dina, Danish Informatics Network in the Agricultural Sciences

grown animals – for instance because of infection risks when animals are bought at the market.

The problem of availability is only one example of a restraint that applies to the herd as a whole. Other examples might be a milk quota, a limited supply of roughages or limiting housing capacity. In all cases the animals considered for replacement compete for the resource (or quota) in question. We shall therefore in this study consider the more general problem of optimal replacement under some *herd restraint*.

The main reason for using Markov decision programming in the determination of optimal animal replacement policies is probably the variation in traits, which with this technique is taken into account directly. Also the cyclic production may be directly considered by traditional Markov decision programming. Very soon, however, a problem of *dimensionality* is faced. If *several* traits of the animal are considered simultaneously, and each trait is considered at a realistic number of levels, the *state space* becomes very large (the size of the state space is in principle calculated as the number of traits times the number of levels of each). Even though the method in theory can handle the problem, optimization is prohibitive even on modern computers. In literature, the problem is referred to as the “*curse of dimensionality*”.

The objective of this study is to discuss how the technique (Markov decision programming) may be adapted to solve the problem (the animal replacement problem), where *uniformity* and *herd restraints* as well as the *curse of dimensionality* (arising from the variability in traits and the cyclic production) have been identified as major difficulties to be taken into account. During the decade since the reviews of Kennedy (1981) and van Arendonk (1984) were written, many results have been achieved concerning these difficulties.

We shall assume throughout the study that the objective of the farmer is the maximization of net revenue from the *entire* herd. In each situation, we shall consider how this objective may be transformed to a relevant criterion of optimality to be used in the Markov decision process.

## 2. Variability and cyclic production: Markov decision programming

As mentioned in the introduction, Markov decision programming is directly able to take the variability in traits and the cyclic production into account without any adaptations. In order to have a frame of reference, we shall briefly present the theory of traditional Markov decision programming originally described by Howard (1960).

### 2.1. Notation and terminology

Consider a discrete time Markov decision process with a finite *state space*  $U = \{1, 2, \dots, u\}$  and a finite *action set*  $D$ . A *policy*  $s$  is a map assigning to each state  $i$  an action  $s(i) \in D$ . Let  $p_{ij}^d$  be the *transition probability* from state  $i$  to state  $j$  if the action  $d \in D$  is taken. The *reward* to be gained when the state  $i$  is observed, and the action  $d$  is taken, is denoted as  $r_i^d$ . The time interval between two transitions is called a *stage*.

We have now defined the elements of a traditional Markov decision process, but in some models we further assume that if state  $i$  is observed, and action  $d$  is taken, a physical quantity of  $m_i^d$  is involved (e.g. Kristensen, 1989; 1991). In this study we shall refer to  $m_i^d$  as the *physical output*. If  $s(i) = d$ , the symbols  $r_i^d$ ,  $m_i^d$  and  $p_{ij}^d$  are also written as  $r_i^s$ ,  $m_i^s$  and  $p_{ij}^s$ , respectively.

An *optimal policy* is defined as a policy that maximizes (or minimizes) some predefined objective function. The optimization technique (i.e. the method to identify an optimal policy) depends on the form of the objective function or – in other words – on the criterion of optimality. The over-all objective to maximize net revenue of the entire herd may (depending on the circumstances) result in different criteria of optimality formulated as alternative objective functions. The choice of criterion depends on whether the planning horizon is finite or infinite.

### 2.2. Criteria of optimality

#### 2.2.1. Finite planning horizon

A farmer, who knows that he is going to terminate his production after  $N$  stages, may use the maximization of total expected rewards as his criterion

of optimality. The corresponding objective function  $h$  is

$$h(s^1, \dots, s^N) = E\left(\sum_{n=1}^N r_{I(n)}^{s^n}\right), \quad (1)$$

where  $E$  denotes the expected value,  $s^n$  is the policy at stage  $n$ , and  $I(n)$  is the (unknown) state at stage  $n$ .

If the farmer has a time preference, so that he prefers an immediate reward to an identical reward later on, a better criterion is the maximization of total expected discounted rewards. If all stages are of equal length, this is equal to applying the objective function

$$h(s^1, \dots, s^N) = E\left(\sum_{n=1}^N \beta^{n-1} r_{I(n)}^{s^n}\right), \quad (2)$$

where  $\beta < 1$  is the discount factor defined by the interest rate and the stage length.

### 2.2.2. Infinite planning horizon

A situation where the stage of termination is unknown (but at least far ahead) is usually modeled by an infinite planning horizon (i.e.  $N = \infty$ ). In that case the optimal policy is constant over stages. The function (1) cannot be applied in this situation, but since  $\beta < 1$ , the function (2) will converge towards a fixed value for  $N$  becoming very large. Thus the objective function is given by

$$h(s) = E\left(\sum_{n=1}^{\infty} \beta^{n-1} r_{I(n)}^s\right). \quad (3)$$

Since, usually, each animal and its future successors are represented by a separate Markov decision process, this criterion together with the criterion (2), are equal to the maximization of total discounted net revenues *per animal*. Such a criterion is relevant in a situation where a limiting housing capacity is the only (or at least the *most* limiting) herd restraint.

An alternative criterion under infinite planning horizon is the maximization of expected average reward per unit of time. If all stages are of equal length, the objective function in this situation is

$$h(s) = g^s = \sum_{i=1}^u \pi_i^s r_i^s, \quad (4)$$

where  $\pi_i^s$  is the limiting state probability under the policy  $s$  (i.e. when the policy is kept constant over an infinite number of stages). This criterion maximizes the average net revenues per stage, i.e. *over time*. It may be relevant under the same conditions as criterion (3) if an animal and its future successors are represented by a separate Markov decision process. Practical experience shows that the optimal replacement policies determined under criteria (3) and (4) are almost identical.

If a herd restraint (e.g. a milk quota) is imposed on the physical output, a relevant criterion may be the maximization of expected average reward per unit of physical output using the objective function

$$h(s) = g^s = \frac{g_r^s}{g_m^s} = \frac{\sum_{i=1}^u \pi_i^s r_i^s}{\sum_{i=1}^u \pi_i^s m_i^s}. \quad (5)$$

In case of a milk quota, the physical output  $m_i^s$  is the milk produced by a cow in state  $i$  under policy  $s$ . The function (5) is also relevant if the criterion is the maximization of the expected average reward over time in a model where the stage length varies. In that case the physical output represents the stage length. It should be noticed that if  $m_i^d = 1$  for all  $i$  and  $d$ , the function (5) is identical to (4). The symbol  $g_r^s$  is the average reward over stages (equal to  $g^s$  of Eq. (4)) and  $g_m^s$  is the average physical output over stages.

## 2.3. Optimization techniques in general Markov decision programming

### 2.3.1. Value iteration

Under finite planning horizon the *value iteration method* is excellent. The optimal policies are determined sequentially using the functional equations

$$f_i(n) = \max_d \left\{ r_i^d + \beta \sum_{j=1}^u p_{ij}^d f_j(n-1) \right\}, \quad i=1, \dots, u, \quad (6)$$

where the action  $d$  maximizing the right hand side is optimal for state  $i$  at the stage in question. The function  $f_i(n)$  is the total expected discounted rewards from the process when it starts from state  $i$  and will operate for  $n$  stages before termination. Thus  $f_i(0)$  is the salvage value of the system when it is in state  $i$ . At each stage an optimal policy is

chosen using Eqs. (6). If the objective function (1) is used,  $\beta = 1$  in Eq. (6). Otherwise  $\beta$  is the discount factor.

Under infinite planning horizon, the value iteration method may be used to approximate an optimal policy. Under the objective function (3) it is possible to show that (cf. Howard 1960)

$$\lim_{n \rightarrow \infty} f_i(n) = f_i, \quad i=1, \dots, u, \quad (7)$$

where  $f_i$  for fixed  $i$  is a constant. By using Eqs. (6) over a large number of stages, we will sooner or later observe that  $f_i(n+1)$  is almost equal to  $f_i(n)$  for all  $i$ . Further we will observe that the same policy is chosen during several stages. We can feel rather sure that such a policy is close to be optimal, but there is no guarantee that it is identical to an optimal policy. For practical purposes, however, the approximation usually suffices.

Since the objective function (4) is just a special case of function (5), where  $m_i^s = 1$  for all  $i$  and  $d$ , we shall only consider the criterion given by (5). In this case  $f_i(n)$  is the total expected rewards when the process starts from the beginning of a stage in state  $i$  and will operate until  $n$  units of physical output have been produced. Under the criterion given by the objective function (4), the production of  $n$  units of output is just the operation of the process over  $n$  stages. It is assumed that the physical output only takes integer values (for practical purpose this is just a question of selecting an appropriate unit). According to Howard (1971) an optimal policy for producing  $n$  units of output (i.e. a policy that maximizes the expected reward of producing  $n$  units) is determined recursively by the relations ( $i=1, \dots, u$ ):

$$f_i(n) = \max_d \left\{ a(n r_i^d / m_i^d + f_i(0)) + (1-a) \left( r_i^d + \sum_{j=1}^u p_{ij}^d f_j(n - m_i^d) \right) \right\}, \quad n=1, \dots$$

where

$$a = \begin{cases} 1, & m_i^d \geq n \\ 0, & m_i^d < n \end{cases}.$$

This is under the assumption that the reward/output rate has the constant value of  $r_i^d/m_i^d$  during the entire stage. However, since the physical output is bounded, it is easily seen that for  $n$  sufficiently large,  $a = 0$ . Hence we get for  $i=1, \dots, u$

$$f_i(n) = \max_d \left\{ r_i^d + \sum_{j=1}^u p_{ij}^d f_j(n - m_i^d) \right\}, \quad \text{large } n. \quad (8)$$

Thus in the long run, the assumption concerning constant reward/output rate in all states will have no effect. The equivalence of Eq. (7) is in this case

$$\lim_{n \rightarrow \infty} (f_i(n) - f_i(n-1)) = g, \quad (9)$$

and sooner or later the policy will not differ from step to step of Eqs. (8).

Further details on the value iteration method are given by Howard (1960; 1971). It should particularly be noticed that  $m_i^d$ , which in this study is interpreted as a physical output (e.g. milk yield), in the study of Howard (1971) is interpreted as the expected stage length when state  $i$  is observed under the action  $d$ . Thus in his model the criterion (5) is the expected average reward over time. Compared to Eq. (8), Howard (1971) described a more general case where the stage length is a random variable of which the distribution is given by the action and the present state as well as the state to be observed at the next stage. Further the reward depends on the state combination, the action and the stage length. The interpretation as physical output has been discussed by Kristensen (1991).

The value iteration method is identical to what is usually referred to as *dynamic programming*, *successive iteration* or *successive approximation*.

### 2.3.2. Policy iteration

Under infinite planning horizon, the *policy iteration method* may be applied. Unlike the value iteration method it always provides an optimal policy. It covers all three objective functions (3), (4) and (5). The iteration cycle used for optimization has the following steps:

- 1) Choose an arbitrary policy  $s$ . Go to 2.
- 2) Solve the set of linear simultaneous equations appearing in Table 1. Go to 3.
- 3) For each state  $i$ , find the action  $d'$  that maximizes the expression given in Table 1, and put  $s'(i)=d'$ . If  $s'=s$  then stop, since an optimal policy is found. Otherwise redefine  $s$  according to the new policy (i.e. put  $s=s'$ ) and go back to 2.

Table 1. Equations and expressions to be used in the policy iteration cycle with different objective functions.

Objective function	Linear equations of Step 2			Expression Step 3
	Equation (i=1,...,u)	Unknowns	Additional equation	
(3)	$f_i^s = r_i^s + \beta \sum_{j=1}^u p_{ij}^s f_j^s$	$f_1^s, \dots, f_u^s$	—	$r_i^d + \beta \sum_{j=1}^u p_{ij}^d f_j^s$
(4)	$g^s + f_i^s = r_i^s + \sum_{j=1}^u p_{ij}^s f_j^s$	$g^s, f_1^s, \dots, f_u^s$	$f_u^s = 0$	$r_i^d + \sum_{j=1}^u p_{ij}^d f_j^s$
(5)	$g^s m_i^s + f_i^s = r_i^s + \sum_{j=1}^u p_{ij}^s f_j^s$	$g^s, f_1^s, \dots, f_u^s$	$f_u^s = 0$	$r_i^d - m_i^d g^s + \sum_{j=1}^u p_{ij}^d f_j^s$

From the equations and expressions of Table 1, we see that also with the policy iteration method the objective function (4) is just a special case of (5) where  $m_i^s = 1$  for all  $i$  and  $d$ . For the objective functions (3) and (4) the policy iteration method was developed by Howard (1960), and for the function (5) a policy iteration method was presented by Jewell (1963). Like Howard (1971), Jewell interpreted  $m_i^d$  as the expected stage length.

Under Criterion (3),  $f_i^s$  is the total present value of the expected future rewards of a process starting in state  $i$  and running over an infinite number of stages following the constant policy  $s$ . Under Criteria (4) and (5),  $f_i^s$  is the *relative value* of state  $i$  under the policy  $s$ . The difference in relative values between two states equals the amount of money a rational person is just willing to pay in order to start in the highest ranking of the two states instead of the lowest ranking. The absolute value of  $f_i^s$  is determined arbitrarily by the additional equation of Table 1, where the relative value of state  $u$  is defined to be zero. The interpretation of relative values is discussed in details by Kristensen (1991).

### 2.3.3. Linear programming

Under an infinite planning horizon, linear programming is a possible optimization technique. When the criterion (3) is applied the linear programming problem becomes (cf. Ross, 1970)

$$\sum_{i=1}^u x_i = \text{Max!}$$

subject to (10)

$$x_i - \beta \sum_{j=1}^u p_{ij}^d x_j \geq r_i^d, \quad \text{all } d \in D, \quad i=1, \dots, u.$$

It appears from (10) that each combination of state and action is represented by exactly one restriction. An action  $d$  is optimal in state  $i$  if, and only if, the corresponding restriction is satisfied as an equation when the values of  $x_1, \dots, x_u$  arises from an optimal solution to the linear programming problem. The *optimal* values of  $x_1, \dots, x_u$  are equal to the present values  $f_1^s, \dots, f_u^s$  under an optimal policy.

If the objective function (4) is applied, the linear programming problem becomes

$$\sum_{i=1}^u \sum_{d \in D} r_i^d x_i^d = \text{Max!}$$

subject to

$$\sum_{d \in D} x_i^d - \sum_{j=1}^u \sum_{d \in D} p_{ij}^d x_j^d = 0, \quad i=1, \dots, u \quad (11)$$

$$\sum_{i=1}^u \sum_{d \in D} x_i^d = 1$$

$$x_i^d \geq 0, \quad d \in D, \quad i=1, \dots, u.$$

In this case an action  $d$  is optimal in state  $i$  if and only if  $x_i^d$  from the optimal solution is strictly positive. The optimal value of the objective function is equal to the average rewards per stage under an optimal policy. The optimal value of  $\sum_{d \in D} x_i^d$  is

equal to the limiting state probability  $\pi_i$  under an optimal policy.

Using Criterion (5), we may solve the following linear programming problem (cf. Kennedy, 1986):

$$\begin{aligned}
 & x_u = \text{Max!} \\
 & \text{subject to} \\
 & -x_i + \sum_{j=1}^{u-1} p_{ij}^d x_j - m_i^d x_u \leq -r_i^d - p_{iu}^d a, \quad (12) \\
 & d \in D, i=1, \dots, u-1 \\
 & \sum_{j=1}^{u-1} p_{uj}^d x_j - m_u^d x_u \leq -r_u^d - p_{uu}^d a + a, \quad d \in D \\
 & x_i \geq 0, \quad i=1, \dots, u,
 \end{aligned}$$

where  $a$  is a pre-determined relative value of state  $u$  chosen sufficiently large to ensure that all other relative values are positive. The optimal value of the objective function of the linear programming problem is equal to the expected average reward per unit of output as defined in Eq. (5) under an optimal policy. The optimal values of the variables  $x_1, \dots, x_{u-1}$  are equal to the relative values of the states  $1, \dots, u-1$ , provided that the relative value of state  $u$  is equal to  $a$ . As it appears, each combination of state and action is represented by one and only one restriction. An action is optimal in a state if and only if the corresponding restriction is satisfied as an equation in the optimal solution.

Since Criterion (4) is just a special case of (5) with all physical outputs set to the value 1, the linear programming problem (12) may also be used in the determination of an optimal policy under Criterion (4).

## 2.4. Discussion and applications

Under finite planning horizon, the value iteration method is perfect, but in replacement models the planning horizon is rarely well defined. Most often the process is assumed to operate over an unknown period of time with no pre-determined stage of termination. In such cases the abstraction of an infinite planning horizon seems more relevant. Therefore we shall pay specific attention to the optimization problem under the criteria (3), (4) and (5) where all three techniques described in the previous sections are available.

The value iteration method is not exact, and the

convergence is rather slow. On the other hand, the mathematical formulation is very simple, and the method makes it possible to handle very large models with thousands of states. Further it is possible to let the reward and/or the physical output depend on the stage number in some pre-defined way. This has been mentioned by van Arendonk (1984) as an advantage in modelling genetic improvement over time. The method has been used in a lot of dairy cow replacement models as an approximation to the infinite stage optimum. Thus it has been used by Jenkins and Halter (1963), Giaever (1966), Smith (1971), McArthur (1973), Steward et al. (1977; 1978), Killen and Kearney (1978), Ben-Ari et al. (1983), van Arendonk (1985; 1986) and van Arendonk and Dijkhuizen (1985). Some of the models mentioned have been very large. For instance, the model of van Arendonk and Dijkhuizen contained 174 000 states (reported by van Arendonk, 1988). In sows, the method has been used by Huirne et al. (1988).

The policy iteration method has almost exactly the opposite characteristics of the value iteration method. Because of the more complicated mathematical formulation involving solution of large systems of simultaneous linear equations, the method can only handle rather small models with, say, a few hundred states. The solution of the linear equations implies the inversion of a matrix of the dimension  $u \times u$ , which is rather complicated. On the other hand, the method is exact and very efficient in the sense of fast convergence. The rewards are not allowed to depend on the stage except for a fixed rate of annual increase (e.g. inflation) or decrease. However, a seasonal variation in rewards or physical outputs is easily modeled by including a state variable describing season (each state is usually defined by the value of a number of state variables describing the system).

An advantage of the policy iteration method is that the equations in Table 1 are *general*. Under any policy  $s$  we are able to calculate directly the economic consequences of following the policy by solution of the equations. This makes it possible to compare the economic consequences of various non-optimal policies to those of the optimal. Further we may use the equations belonging to the criterion (5) to calculate the long run technical results under a given policy by redefining  $r_i^s$  and  $m_i^s$ . If for instance  $r_i^s = 1$  if a calving takes place and zero otherwise, and  $m_i^s$  is the stage length when

state  $i$  is observed under policy  $s$ , then  $g^s$ , which is the average number of calvings per cow per year, may be determined from the equations. Further examples are discussed by Kristensen (1991). For an example where the equations are used for calculation of the economic value of culling information, reference is made to Kristensen and Thysen (1991).

The policy iteration method has been used by Reenberg (1979) and Kristensen and Østergaard (1982). The models were very small, containing only 9 and 177 states, respectively.

### 3. The curse of dimensionality: Hierarchic Markov processes

In order to combine the computational advantages of the value iteration method with the exactness and efficiency of the policy iteration method Kristensen (1988; 1991) introduced a new notion of a hierarchic Markov process. It is a contribution to the solution of the problem referred to as the "curse of dimensionality" since it makes it possible to give exact solutions to models with even very large state spaces. A hierarchic Markov process is only relevant under infinite planning horizon, and there is no relevance of the criterion (4) because the special situation where the physical output equals 1 in all stages has no computational advantages over other values. Therefore we shall only consider the criteria (3) and (5).

#### 3.1. Notation and terminology

A hierarchic Markov process is a series of Markov decision processes called *subprocesses* built together in one Markov decision process called the *main process*. A subprocess is a finite time Markov decision process with  $N$  stages and a finite state space  $\Omega_n = \{1, \dots, u_n\}$  for stage  $n$ ,  $1 \leq n \leq N$ . The action set  $\mathbf{D}_n$  of the  $n$ th stage is assumed to be finite, too. A policy  $s$  of a subprocess is a map assigning to each stage  $n$  and state  $i \in \Omega_n$  an action  $s(n, i) \in \mathbf{D}_n$ . The set of all possible policies of a subprocess is denoted  $\Gamma$ . When the state  $i$  is observed and the action  $d$  is taken, a reward  $r_i^d(n)$  is gained. The corresponding physical output is denoted as  $m_i^d(n)$ . Let  $p_{ij}^s(n)$  be the transition probability from state  $i$  to state  $j$  where  $i$  is the state at the  $n$ th stage,  $j$  is the state at the following stage

and  $d$  is the action taken at stage  $n$ . Under the Criterion (3) we shall denote the discount factor in state  $i$  under the action  $d$  as  $\beta_i^d(n)$  assuming that the stage length is given by stage, state and action.

Assume that we have a set of  $v$  possible subprocesses each having its own individual set of parameters. The main process is then a Markov decision process running over an infinite number of stages and having the finite state space  $\{1, \dots, v\}$ . Each stage in this process represents a particular subprocess. The action sets of the main process are the sets  $\Gamma_\iota$ ,  $\iota = 1, \dots, v$ , of all possible policies of the individual subprocesses (to avoid ambiguity the states of the main process will be denoted by Greek letters  $\iota$ ,  $\kappa$  etc.). A policy  $\sigma$  is a map assigning to each state  $\iota$  of the main process an action  $\sigma(\iota) \in \Gamma_\iota$ . The transition matrix of the main process has the dimension  $v \times v$ , and it is denoted  $\Phi = \{\phi_{\iota\kappa}\}$ . The transition probabilities are assumed to be independent of the action taken. The reward  $f_\iota^\sigma$  and the physical output  $h_\iota^\sigma$  in state  $\iota$  of the main process are determined from the total rewards and output functions of the corresponding subprocess

$$f_i^s(n) = r_i^s(n), \quad n=N$$

$$f_i^s(n) = r_i^s(n) + \beta_i^s(n) \sum_{j=1}^{u_{n+1}} p_{ij}^s(n) f_j^s(n+1), \quad n=1, \dots, N-1, \quad (13)$$

and

$$f_\iota^\sigma = \sum_{i=1}^{u_1} p_i(0) f_i^s(1), \quad s = \sigma(\iota),$$

and analogously for  $h_\iota^\sigma$  (except for the discount factor). The symbol  $p_i(0)$  is the probability of observing state  $i$  at the first stage of the subprocess. Finally, the expected discount factor in state  $\iota$  under the action  $s$  is denoted as  $B_\iota^s$  and calculated as follows

$$b_i^s(n) = \beta_i^s(n), \quad n=N$$

$$b_i^s(n) = \beta_i^s(n) \sum_{j=1}^{u_{n+1}} p_{ij}^s(n) b_j^s(n+1), \quad n=1, \dots, N-1, \quad (14)$$

and

$$B_\iota^s = \sum_{i=1}^{u_1} p_i(0) b_i^s(1).$$

### 3.2. Optimization

Since the main process is just an ordinary Markov decision process, the policy iteration cycle described in Section 2.3.2 might be used directly for optimization. In practice Steps 1 and 2 are easily carried out, but Step 3 is prohibitive because of the extremely large number of alternative actions  $s \in \Gamma_i$  (as mentioned above  $s$  is an entire policy of the  $i$ th subprocess). To circumvent this problem Kristensen (1988; 1991) constructed an iterative method, where a value iteration method is applied in the subprocesses and the results are used in Step 3 of the policy iteration method of the main process. The different versions of the method covers the criteria of optimality under infinite planning horizon defined as (3) and (5) in Section 2.2.2. Since criterion (4) is a special case of (5) it is also indirectly covered.

The general form of the iteration cycle of a hierarchic Markov process has the following steps:

- 1) Choose an arbitrary policy  $\sigma$ . Go to 2.
- 2) Solve the following set of linear simultaneous equations for  $F_1^\sigma, \dots, F_v^\sigma$  and in case of Criterion (5) for  $g^\sigma$ :

$$g^\sigma h_\nu^\sigma + F_\nu^\sigma = f_\nu^\sigma + B_\nu^\sigma \sum_{\kappa=1}^v \phi_{\nu\kappa} F_\kappa^\sigma, \quad \nu=1, \dots, v.$$

In case of Criterion (5) the additional equation  $F_v^\sigma = 0$  is necessary in order to determine a unique solution. Go to 3.

- 3) Define

$$T_\nu = \sum_{\kappa=1}^v \phi_{\nu\kappa} F_\kappa^\sigma$$

under Criterion (3) and  $T_\nu = 0$  under Criterion (5). For each subprocess  $\nu$ , find by means of the recurrence equations

$$\tau_{\nu,i}(n) = \max_d \left\{ r_i^d(n) - m_i^d(n)g^\sigma + \beta_i^d(N)T_\nu \right\}, \quad n=N$$

$$\tau_{\nu,i}(n) = \max_d \left\{ r_i^d(n) - m_i^d(n)g^\sigma + \beta_i^d(n) \sum_{j=1}^{u_{n+1}} p_{ij}^d(n) \tau_{\nu,j}(n+1) \right\}, \quad n=1, \dots, N-1.$$

a policy  $s'$  of the subprocess. The action  $s'(n, i)$  is equal to the  $d'$  that maximizes the right hand side of the recurrence equation of state  $i$  at stage  $n$ . Put  $\sigma'(\nu) = s'$  for  $\nu=1, \dots, v$ . If  $\sigma' = \sigma$ , then

stop since an optimal policy is found. Otherwise, redefine  $\sigma$  according to the new policy (i.e. put  $\sigma = \sigma'$ ) and go back to 2.

When the iteration cycle is used under Criterion (3) all physical outputs ( $m_i^d(n)$  and accordingly also  $h_i^\sigma$ ) are put equal to zero. The iteration cycle covering this situation was developed by Kristensen (1988).

Under Criterion (4) all physical outputs  $m_i^d(n)$  and all discount factors  $\beta_i^d(n)$  and  $B_i^\sigma$  are put equal to 1, but under Criterion (5) only the discount factors are put equal to 1. The iteration cycle covering these situations was described by Kristensen (1991).

### 3.3. Discussion and applications

The hierarchic Markov process is specially designed to fit the structure of replacement problems where the successive stages of the subprocesses correspond to the age of the asset in question. By appropriate selection of state spaces in the subprocesses and the main process it is possible to find optimal solutions to even very large models. The idea is to let the number of states in the subprocesses (where a value iteration technique is applied) be very large and only include very few states in the main process (where the technique is directly based on the policy iteration method). Thus we have got a method which is at the same time fast, exact and able to handle very large models.

Kristensen (1987) used the technique in a dairy cow replacement model which in a traditional formulation as an ordinary Markov decision process would have contained approximately 60 000 states, and later (Kristensen, 1989) in a model with approximately 180 000 states. In both cases the number of states in the main process was only 5, reducing Step 2 to the solution of only 5 simultaneous linear equations (versus 180 000 in a traditional formulation). Even in these very large models the number of iterations needed to provide an optimal solution was only from 3 to 6 (tested under 100 different price and production conditions, Kristensen, 1991). Recently, the method is applied by Houben et al. (1992).

In sows, Huirne et al. (1992) seem to have applied a technique which in many aspects is similar to a hierarchic Markov process, but they have not explained their method in all details. Also Jør-

gensen (1992a) has applied a technique which is inspired of a hierarchic Markov process in a sow replacement model, and recently (Jørgensen 1992b), he used the hierarchic method in the determination of optimal delivery policies in slaughter pigs.

Naturally the hierarchic model just described may also be formulated as an ordinary Markov decision process. In that case each combination of subprocess (main state), stage and state should be interpreted as a state. We shall denote a state in the transformed process as  $(uni)$ , and the parameters are

$$\begin{aligned} r_{uni}^d &= r_i^d(n), \\ m_{uni}^d &= m_i^d(n), \\ \beta_{uni}^d &= \beta_i^d(n), \\ p_{(uni)(\kappa m j)}^d &= \begin{cases} p_{ij}^d(n), & \iota = \kappa \wedge m = n-1 \\ \phi_{\iota \kappa} p_i(0), & n = N \wedge m = 1 \\ 0, & \text{otherwise} \end{cases}, \end{aligned} \quad (15)$$

where the parameters mentioned on the right hand side of the equations are those belonging to the  $\iota$ th subprocess except for  $p_i(0)$  which belongs to subprocess  $\kappa$ . This formulation of course has the same optimal policies as the hierarchic formulation, so it is only computational advantages that make the hierarchic model relevant. A comparison to traditional methods may therefore be relevant.

Since the policy iteration method involves the solution of a set of  $u$  equations (where  $u$  is the number of states) it is only relevant for small models. The value iteration method, however, has been used with even very large models and may handle problems of the same size as the hierarchic formulation, but the time spent on optimization is much lower under the hierarchic formulation. To recognize this, we shall compare the calculations involved.

Step 3 of the hierarchic optimization involves exactly the same number of operations as one iteration of the value iteration method (Eq. (6)). The further needs of the hierarchic method are the calculation of the rewards and **either** the physical output **or** the expected discount factor of a stage in the main process according to Eqs. (13) and (14). Since the calculations at each stage is only carried out for one action, the calculation of both main state parameters involves approximately the same

number of operations as one iteration under the value iteration method if the number of alternative actions is 2. If the number of actions is higher, the calculations relatively involves a lower number of operations than an iteration under the value iteration method. These considerations are based on the assumption that the value iteration method is programmed in an efficient way, so that the sum of Eq. (6) is not calculated as a sum of all  $u$  elements, but only as a sum of those elements where  $p_{ij}^d$  is not zero according to Eq. (15). Otherwise the hierarchic technique will be even more superior. Finally the system of linear equations of Step 2 of the hierarchic cycle must be solved, but in large models with only a few states in the main process the time spent on this is negligible.

If we use the considerations above in a practical example, the advantages of the hierarchic technique becomes obvious. As reported by Kristensen (1991) a model with 180 000 state combinations was optimized by the hierarchic technique under 100 different price conditions. The number of iterations needed ranged from 3 to 6 corresponding to between 6 and 12 iterations of the value iteration method. If the latter method was used instead, a planning horizon of 20 years would be realistic (cf. van Arendonk 1985). Since each state in the model equals 4 weeks, this horizon represents 260 iterations, which should be compared to the equivalence of from 6 to 12 when the hierarchic technique was applied.

### 3.4. A numerical example of a hierarchic Markov process

Consider an asset (e.g. a dairy cow) producing two kinds of output items (1 and 2, e.g. milk and beef). We shall assume that the production level of item 1 may change stochastically over time, whereas the production of item 2 is constant over the entire life time of the asset (but may vary between individual assets). At regular time intervals (stages) the asset is inspected in order to determine the production level of item 1. At the first inspection of the asset the production level of item 2 is also determined. In both cases we assume that the result may be "bad", "normal" or "good" (representing the production of 5, 6 and 7 units of item 1 or 3, 4 and 5 units of item 2). After inspection we can choose to keep the asset for at least one additional stage, or

Table 2. Parameters of the hierarchic Markov process, subprocesses.

Sub- pr. $\tau$	Stg. $n$	St. $i$	$p_{ij}^1(n)$				$m_i^1(n)$	$r_i^1(n)$	$p_{ij}^2(n)$				$m_i^2(n)$	$r_i^2(n)$
			j=1	j=2	j=3	j=4			j=1	j=2	j=3	j=4		
1	1	1	0.6	0.3	0.1	0.0	5	7	0.0	0.0	0.0	1.0	5	5
1	1	2	0.2	0.6	0.2	0.0	6	8	0.0	0.0	0.0	1.0	6	6
1	1	3	0.1	0.3	0.6	0.0	7	9	0.0	0.0	0.0	1.0	7	7
1	1	4	0.0	0.0	0.0	1.0	0	0	0.0	0.0	0.0	1.0	0	0
1	2	1	0.6	0.3	0.1	0.0	5	6	0.0	0.0	0.0	1.0	5	4
1	2	2	0.2	0.6	0.2	0.0	6	7	0.0	0.0	0.0	1.0	6	5
1	2	3	0.1	0.3	0.6	0.0	7	8	0.0	0.0	0.0	1.0	7	6
1	2	4	0.0	0.0	0.0	1.0	0	0	0.0	0.0	0.0	1.0	0	0
1	3	1	0.6	0.3	0.1	0.0	5	5	0.0	0.0	0.0	1.0	5	3
1	3	2	0.2	0.6	0.2	0.0	6	6	0.0	0.0	0.0	1.0	6	4
1	3	3	0.1	0.3	0.6	0.0	7	7	0.0	0.0	0.0	1.0	7	5
1	3	4	0.0	0.0	0.0	1.0	0	0	0.0	0.0	0.0	1.0	0	0
1	4	1	—	—	—	—	5	2	—	—	—	—	5	2
1	4	2	—	—	—	—	6	3	—	—	—	—	6	3
1	4	3	—	—	—	—	7	4	—	—	—	—	7	4
1	4	4	—	—	—	—	0	0	—	—	—	—	0	0
2	1	1	0.6	0.3	0.1	0.0	5	8	0.0	0.0	0.0	1.0	5	6
2	1	2	0.2	0.6	0.2	0.0	6	9	0.0	0.0	0.0	1.0	6	7
2	1	3	0.1	0.3	0.6	0.0	7	10	0.0	0.0	0.0	1.0	7	8
2	1	4	0.0	0.0	0.0	1.0	0	0	0.0	0.0	0.0	1.0	0	0
2	2	1	0.6	0.3	0.1	0.0	5	7	0.0	0.0	0.0	1.0	5	5
2	2	2	0.2	0.6	0.2	0.0	6	8	0.0	0.0	0.0	1.0	6	6
2	2	3	0.1	0.3	0.6	0.0	7	9	0.0	0.0	0.0	1.0	7	7
2	2	4	0.0	0.0	0.0	1.0	0	0	0.0	0.0	0.0	1.0	0	0
2	3	1	0.6	0.3	0.1	0.0	5	6	0.0	0.0	0.0	1.0	5	4
2	3	2	0.2	0.6	0.2	0.0	6	7	0.0	0.0	0.0	1.0	6	5
2	3	3	0.1	0.3	0.6	0.0	7	8	0.0	0.0	0.0	1.0	7	6
2	3	4	0.0	0.0	0.0	1.0	0	0	0.0	0.0	0.0	1.0	0	0
2	4	1	—	—	—	—	5	3	—	—	—	—	5	3
2	4	2	—	—	—	—	6	4	—	—	—	—	6	4
2	4	3	—	—	—	—	7	5	—	—	—	—	7	5
2	4	4	—	—	—	—	0	0	—	—	—	—	0	0
3	1	1	0.6	0.3	0.1	0.0	5	9	0.0	0.0	0.0	1.0	5	7
3	1	2	0.2	0.6	0.2	0.0	6	10	0.0	0.0	0.0	1.0	6	8
3	1	3	0.2	0.3	0.6	0.0	7	11	0.0	0.0	0.0	1.0	7	9
3	1	4	0.0	0.0	0.0	1.0	0	0	0.0	0.0	0.0	1.0	0	0
3	2	1	0.6	0.3	0.1	0.0	5	8	0.0	0.0	0.0	1.0	5	6
3	2	2	0.2	0.6	0.2	0.0	6	9	0.0	0.0	0.0	1.0	6	7
3	2	3	0.1	0.3	0.6	0.0	7	10	0.0	0.0	0.0	1.0	7	8
3	2	4	0.0	0.0	0.0	1.0	0	0	0.0	0.0	0.0	1.0	0	0
3	3	1	0.6	0.3	0.1	0.0	5	7	0.0	0.0	0.0	1.0	5	5
3	3	2	0.2	0.6	0.2	0.0	6	8	0.0	0.0	0.0	1.0	6	6
3	3	3	0.1	0.3	0.6	0.0	7	9	0.0	0.0	0.0	1.0	7	7
3	3	4	0.0	0.0	0.0	1.0	0	0	0.0	0.0	0.0	1.0	0	0
3	4	1	—	—	—	—	5	4	—	—	—	—	5	4
3	4	2	—	—	—	—	6	5	—	—	—	—	6	5
3	4	3	—	—	—	—	7	6	—	—	—	—	7	6
3	4	4	—	—	—	—	0	0	—	—	—	—	0	0

we can choose to replace it at the end of the stage at some additional cost.

The three classes of production level of item 2 are defined as states in the main process of a hierarchic Markov process. Thus the number of subprocesses is also 3 and each subprocess represents an asset of a certain productivity concerning item 2. When a new asset is purchased, we assume that the probability distribution over main states is uniform, so that the probability of entering either one is  $1/3$ . The maximum age of an asset is assumed to be 4 stages, and the states of the subprocess are defined from the productivity concerning item 1. Further a dummy state of length, reward and output equal to 0 is included at each stage of the subprocesses. If the asset is replaced at the end of a stage, the process enters the dummy state with probability 1 at the next stage, and for the rest of the duration of the subprocess it will stay in the dummy states.

For all subprocesses we assume that, if the asset is kept, the probability of staying at the same productivity level (state in the subprocess) concerning item 1 is 0.6, and if the present state is "normal", the probability of transition to either "bad" or "good" is 0.2 each. The probability of transition (if kept) from "bad" or "good" to "normal" is in both cases 0.3, and from "bad" to "good" and vice versa the probability is 0.1. The initial state probabilities of the subprocesses are assumed to depend on the subprocess in such a way that for subprocess number 1 (low productivity of item 2) the probabilities of entering state "bad", "normal" and "good" are 0.6, 0.3 and 0.1 respectively. For subprocess number 2 the corresponding probabilities are 0.2, 0.6 and 0.2 and finally for subprocess number 3 they are 0.1, 0.3, 0.6.

The physical output  $m_i^d(n)$  of state  $i$  at stage  $n$  of subprocess number  $\iota$  is equal to the production of item 1 under the action  $d$ , and the corresponding rewards are assumed to be defined as follows:

$$r_{\iota,i}^d(n) = c_1 m_i^d + c_2 k_{\iota} - c_n - c_3^d, \quad \iota=1,2,3, \\ n=1,\dots,4, \quad i=1,\dots,4, \quad d=1,2, \quad (16)$$

where  $c_1$  is the price of item 1,  $c_2$  is the price of item 2,  $c_n$  is the cost of operating the asset at the age  $n$ ,  $k_{\iota}$  is the production of item 2 in subprocess (main state) number  $\iota$  and  $c_3$  is the replacement cost which is zero if no replacement takes place. The cost of operating the asset is assumed to increase linearly from 1 to 4 over stages. Defining  $c_1 = c_2 = 1$  and  $c_3^2 = 2$  gives us the final parameters appearing in Tables 2 and 3. All stages (except those where the process is in a dummy state of zero length) are assumed to be of equal length, which we for convenience put equal to 1.

We shall determine an optimal solution under the following 3 criteria of optimality:

- 1) Maximization of total expected discounted rewards, i.e., the objective function (3). In this case the physical outputs of Table 2 are ignored, and a discount factor  $\beta_i^d(n) = \exp(-r)$ , where  $r$  is the interest rate, is applied (for states where the stage length is not zero).
- 2) Maximization of average rewards over time. In this situation we use the objective function (5) letting the output represent stage length. No discounting is performed in this case.
- 3) Maximization of average rewards over output defined as in Table 2. Thus the objective function (5) is applied, and no discounting is performed.

Table 3. Parameters of the hierarchic process. Transition probabilities of main process and initial state probabilities of subprocesses

Transition probabilities, main process				Initial state probabilities, subprocesses			
Main state $\iota$	$\phi_{\iota\kappa}$			$p_i(0)$			
	$\kappa=1$	$\kappa=2$	$\kappa=3$	$i=1$	$i=2$	$i=3$	$i=4$
1	1/3	1/3	1/3	0.6	0.3	0.1	0.0
2	1/3	1/3	1/3	0.2	0.6	0.2	0.0
3	1/3	1/3	1/3	0.1	0.3	0.6	0.0

Table 4. Optimal policies under the three criteria (c1, c2, c3) defined in the text (actions: 1="keep", 2="replace").

Subprocess	Stage	State 1			State 2			State 3		
		c1	c2	c3	c1	c2	c3	c1	c2	c3
1	1	2	2	2	2	2	2	2	2	2
1	2	2	2	2	2	2	2	2	2	2
1	3	2	2	2	2	2	2	2	2	2
2	1	2	2	1	1	1	1	1	1	2
2	2	2	2	2	2	2	2	2	2	2
2	3	2	2	2	2	2	2	2	2	2
3	1	1	1	1	1	1	1	1	1	1
3	2	2	2	1	1	1	1	1	1	2
3	3	2	2	2	2	2	2	2	2	2

In Table 4, optimal policies under the three criteria are shown. It appears that the same policies are optimal under the first two criteria, but under the third criterion the optimal policy differs. A more detailed example of the effect of criterion of optimality was discussed by Kristensen (1991).

In order to compare the efficiency of the hierarchic technique to the traditional policy and value iteration methods, the problem of the example was transformed to an ordinary Markov decision process and optimized by those methods. The transformed model has  $3 \times 4 \times 4 = 48$  states, which is not larger than the policy iteration method may be applied without problems. In Table 5 some performance data of the three optimization techniques are compared.

The superiority of the hierarchic technique over the policy iteration method is due mainly to the time spent on solving the linear simultaneous equations of Step 2. In the hierarchic case a system of 3 equations is solved, whereas 48 equations are solved under the ordinary policy iteration method.

In this numerical example the performance of the hierarchic technique is even more superior to the value iteration method than expected from the theoretical considerations of Section 3.3. In the present case an iteration of the hierarchic model is performed even faster than one of the value iteration method applied to the same (transformed) model. The reason is that the value iteration algorithm has not been programmed in the most efficient way as defined in Section 3.3. On the contrary, the sum of Eq. (6) has been calculated over all 48 states of the transformed model. Since only 4 transition probabilities from each state are positive, the sum could be calculated only over these 4 states.

#### 4. Uniformity: Bayesian updating

As discussed earlier, it is obvious that the traits of an animal varies no matter whether we are considering the milk yield of a dairy cow, the litter size

Table 5. The performance of the hierarchic technique compared to the policy and value iteration methods under the three criteria (c1, c2, c3) defined in the text.

	Hierarchic model			Policy iteration			Value iteration		
	c1	c2	c3	c1	c2	c3	c1	c2	c3
Number of iterations	4	3	3	3	4	3	100	100	100
Computer time, relativity	1	0.82	0.77	120	150	120	62	64	63

of a sow or almost any other trait. On the other hand, it is *not* obvious to what extent the *observed* trait  $Y_n$  at stage  $n$  is, for instant, the result of a permanent property of the animal  $X_1$ , a permanent damage caused by a previous disease  $X_2$  or a temporary random fluctuation  $e_n$ . Most often the observed value is the result of several permanent and random effects. With  $Y_n$ ,  $X_1$ ,  $X_2$  and  $e_n$  defined as above the relation might for instance be

$$Y_n = m + X_1 + aX_2 + e_n, \quad (17)$$

where  $m$  is the expected value for an arbitrarily selected animal under the circumstances in question, and  $a = -1$  if the animal has been suffering from the disease, and  $a = 0$  otherwise. In this example  $X_1$  only varies *among* animals, whereas  $e_n$  also varies *over time* for the same animal. The effect of the damage caused by the disease  $X_2$  is in this example assumed to be constant over time when it has been "switched on". The value of  $X_2$  is a property of the individual disease case (the "severity" of the case).

In a replacement decision it is of course important to know whether the observed value is mainly a result of a permanent effect or it is just the result of a temporary fluctuation. The problem, however, is that only the resulting value  $Y_n$  is observed, whereas the values of  $X_1$ ,  $X_2$  and  $e_n$  are unknown. On the other hand, as observations of  $Y_1$ ,  $Y_2$ ,... are done we are learning something about the value of the permanent effects. Furthermore, we have got a *prior* distribution of  $X_1$  and  $X_2$ , and each time an observation is done, we are able to calculate the *posterior* distribution of  $X_1$  and  $X_2$  by means of the Kalman-filter theory (described for instance by Harrison and Stevens, 1976) if we assume all effects to be normally distributed.

A model as described by Eq. (17) fits very well into the structure of a hierarchic Markov process. Thus we may regard  $Y_n$  as a state variable in a sub-process, and the permanent effects  $X_1$  and  $X_2$  as state variables of the main process. We then face a hierarchic Markov process with *unobservable main state*. Kristensen (1993) discusses this notion in details, and it is shown that under the assumption of normally distributed effects, we only have to keep the present expected values of  $X_1$  and  $X_2$ , the currently observed value of  $Y_n$  and (in this example) the number of stages since the animal was suffering from the disease (if it has been suffering

from the disease at all). The expectations of  $X_1$  and  $X_2$  are sufficient to determine the current posterior distribution of the variables, because the variance is known in advance. Even though the posterior variance decreases as observations are done, the decrease does *not* depend on the *values* of  $Y_1$ ,  $Y_2$ ,... but only on the number of observations done.

In the study of Kristensen (1993), a more general case involving several traits each being influenced by several unobservable effects is sketched, and a numerical example involving only a single trait is given. An example concerning replacement of sows has been given by Jørgensen (1992a). It was demonstrated in both studies that the Bayesian approach in some cases may result in state space reduction without loss of information.

## 5. Herd restraints: Parameter iteration

One of the major difficulties identified in the introduction was *herd restraints*. All the replacement models mentioned in the previous sections have been single-component models, i.e., only one animal is considered at the same time, assuming an unlimited supply of all resources (heifers or gilts for replacement, feed, labour etc) and no production quota. In a multi-component model all animals of a herd are simultaneously considered for replacement. If all animals (components) compete for the same limited resource or quota, the replacement decision concerning an animal does not only depend on the state of that particular animal, but also on the states of the other animals (components) of the herd.

If the *only* (or at least the *most limiting*) herd restraint is a limited housing capacity, the number of animals in production is the scarce resource, and accordingly the relevant criterion of optimality is the maximization of net revenues per animal as it is expressed in the criteria (1), (2), (3) and (4). Thus the optimal replacement policy of the single component model is optimal for the multi-component model too.

If the *only* (or *most limiting*) herd restraint is a milk quota, the situation is much more complicated. Since the most limiting restriction is a fixed amount of milk to produce, the relevant criterion of optimality is now the maximization of average net revenues per kg milk yield as expressed in criterion (5), because a policy that maximizes net

revenues per kg milk will also maximize total net revenues from the herd which was assumed to be the objective of the farmer.

By following a policy which is optimal according to criterion (5) we assure at any time that the cows which produce milk in the cheapest way are kept. Thus the problem of selecting which cows to keep in the long run (and the mutual ranking of cows) is solved, but the problem of determining the optimal number of cows in production at any time is *not* solved. If for instance, it is recognized 2 months before the end of the quota year that the quota is expected to be exceeded by 10 percent, we have to choose whether to reduce the herd size or to keep the cows and pay the penalty. The problem is that both decisions will influence the possibilities of meeting the quota of the next year in an optimal way. To solve this short run quota adjustment problem we need a true multi-component model.

An other example of a herd restraint is a limited supply of heifers. If the dairy farmer only uses home-grown heifers for replacement, the actions concerning individual cows become inter-dependent, and again a multi-component model is needed in order to solve the replacement problem. Ben-Ari and Gal (1986) and later Kristensen (1992) demonstrated that the replacement problem in a dairy herd with cows and a limited supply of home grown heifers may be formulated as a Markov decision process involving millions of states. This multi-component model is based on a usual single-component Markov decision process representing one cow and its future successors. Even though the hierarchic technique has made the solution of even very large models possible, such a model is far too large for optimization in practice. Therefore, the need for an approximate method emerged, and a method called *parameter iteration* was introduced by Ben-Ari and Gal (1986).

The basic idea of the method is to approximate either the present value function  $f_i(n)$  (objective function (3)) or the relative values  $f_i^s$  (objective functions (4) and (5)) by a function  $G$  involving a set of parameters  $a_1, \dots, a_m$  to be determined in such a way that  $G(i, a_1, \dots, a_m) \approx f_i(n)$  or  $G(i, a_1, \dots, a_m) \approx f_i^s$ .

In the implementation of Ben-Ari and Gal (1986) the parameters were determined by an iterative technique involving the solution of sets of simultaneous linear equations generated by simulation.

In a later implementation Kristensen (1992) de-

termined the parameters by ordinary least squares regression on a simulated data set. The basic idea of the implementation is to take advantage from the fact that we are able to determine an optimal solution to the underlying (unrestricted) single-component model. If no herd restraint was present, the present value of the multi-component model would equal the sum of the present values of the individual animals determined from the underlying single-component model. Then it is argued in what way the restraint will logically reduce the (multi-component) present value, and a functional expression having the desired properties is chosen. The parameters of the function are estimated from a simulated data set, and the optimal action for a given (multi-component) state is determined as the one that maximizes the corrected present value. (A state in the multi-component model is defined from the states of the individual animals in the single-component model, and an action defines the replacement decision of each individual animal).

Ben-Ari and Gal (1986) compared the economic consequences of the resulting optimal multi-component policy to a policy defined by dairy farmers, and they showed that the policy from the parameter iteration method was better. Kristensen (1992) compared the optimal multi-component policies to policies from usual single-component models in extensive stochastic simulations and showed that the multi-component policies were superior in situations with shortage of heifers.

The parameter iteration method has been applied under a limited supply of heifers. It seems to be realistic to expect, that the method and the basic principles of Kristensen (1992) may be used under other kinds of herd restraints as for instance the short time adjustment to a milk quota as mentioned above.

## 6. General discussion

In the introduction, the main difficulties of the animal replacement problem were identified as *variability* in traits, *cyclic production*, *uniformity* (the traits are difficult to define and measure) and *herd restraints*. We are now able to conclude that the difficulties of variability and the cyclic production are directly solved by the application of Markov decision programming, but when the variability of several traits are included we face a problem of *di-*

*dimensionality*. The formulation of the notion of a *hierarchic Markov process* contributed to the solution of the dimensionality problem, but did not solve it. The upper limit of number of states to be included has been raised considerably, but not eliminated.

This is for instance clearly illustrated when we formulate *multi-component* herd models in order to deal with herd restraints. In that case we still have to use approximate methods to determine an "optimal" replacement policy. On the other hand it has been demonstrated by Kristensen (1992) that the *parameter iteration* method applied to a multi-component herd model (even though it is only approximate) is able to improve the total net revenue compared to the application of a usual single-component (animal) model in a situation with shortage of heifers. The parameter iteration method is an important contribution to the problem of determining optimal replacement policies under herd restraints.

In other situations with a limiting herd restraint it may be relevant to use an alternative criterion of optimality maximizing average net revenue per unit of the limiting factor. This method has been successfully applied in a situation with milk production under a limiting quota.

Recent results have also contributed to the solution of the uniformity problem. The Bayesian updating technique described in Section 4 seems to be a promising approach, but it has not yet been tested on real data. It might be a solution to the problem of including animal health as a trait to be considered. The problem of including diseases in the state space has never been solved, but at present Houben et al. (1992) are working on it. As concerns other traits such as litter size or milk yield the Bayesian approach may in some cases result in a reduction of the state space without loss of information (Jørgensen, 1992a; Kristensen, 1993). Thus it contributes indirectly to the solution of the dimensionality problem.

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## Theory and Methodology

# Hierarchic Markov processes and their applications in replacement models

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**Abstract:** In this paper a new notion of a hierarchic Markov process is introduced. It is a series of Markov decision processes called subprocesses built together in one Markov decision process called the main process. The hierarchic structure is specially designed to fit replacement models which in the traditional formulation as ordinary Markov decision processes are usually very large. The basic theory of hierarchic Markov processes is described and examples are given of applications in replacement models. The theory can be extended to fit a situation where the replacement decision depends on the quality of the new asset available for replacement.

**Keywords:** Management, agriculture, optimization, Markov decision programming, stochastic processes

### 1. Introduction

A pioneer of replacement theory was Preinrich [12] who pointed out that every reinvestment in industrial equipment is a link in a chain comprising the permanent occurrence of new, future replacements. His model is usually denoted the ‘*constant chain*’ approach, because he assumed that each link in the replacement chain was equal. Terborgh [18] presented a model which enabled him to take technological change into account, and later [19] he extended his model to consider taxes. Smith [15] relaxed the constant chain assumption of Preinrich by assuming that operating

receipts depend on the purchase time of the new investment.

A common feature of the above mentioned models is that they give a mathematical formulation of the replacement problem assuming complete knowledge of all functions and variables involved. From the mathematical model *general* rules for optimal replacement under some specified restrictions (e.g. the constant chain assumption) are deduced. Risk and uncertainty are not considered.

The introduction of *Dynamic Programming* by Bellman [1] provided a fundamentally different approach. Instead of general rules such models give numerical solutions to concrete replacement problems (and a lot of other problems too). Dynamic programming is very useful when the stochastic elements are introduced in replacement problems. In particular Howard [5] has contributed to the progress in this field with his book on ‘*Dynamic Programming and Markov Processes*’, where he also introduced the so-called *policy itera-*

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tion method to solve multi-stage decision problems in connection with Markov processes.

A more extensive survey of general replacement models is given by Rapp [13].

In the present paper a new notion of a *hierarchic Markov process* is described. It is a series of Markov decision processes called *subprocesses* built together in one Markov decision process called the *main process* so that each stage in the main process represents a subprocess. The concept of a hierarchic Markov process was originally introduced by Kristensen [7] in order to study a dairy cow replacement problem. The theory is, however, relevant for replacement problems in general. We shall describe the general theory, but for examples we shall mostly refer to the dairy cow replacement problem, because this case reveals many features which illustrate the usefulness of the theory. For a larger example concerning the application of a hierarchic Markov process in this field, reference is made to Kristensen [9].

When the theory is applied to a replacement problem it provides, like related dynamic programming techniques, an optimization algorithm to give a *numerical* solution. No general replacement rules are deduced from the theory.

The usual tool for optimization in dairy cow replacement studies is dynamic programming and so in particular Markov decision processes. A review of such studies is given by van Arendonk [20]. Recent papers [21, 22 and 23] also deal with optimization by use of dynamic programming. Ben-Ari and Gal [2] have described a model taking the limited supply of replacement heifers directly into account. They applied a special technique called parameter iteration to approximate an optimal solution.

In most studies [4,6,11,14,16,17,21,22,23] the value iteration method described by Howard [5] is used. This method does not give an exact solution under an infinite planning horizon. However, it is possible to optimize models with a very large number of states (29,880 states were handled by van Arendonk [21]). Kristensen and Østergaard [10] used the policy iteration method. From a theoretical point of view this method has the advantage that it is very efficient and gives an exact solution. A problem is, however, that the method uses inversion of the transition matrix which has the dimension  $n \times n$  where  $n$  is the number of states. In the present paper this prob-

lem is circumvented by the introduction of a hierarchic Markov process. The structure of such a process is like a Markov decision process (main process) where each stage itself represents a Markov decision process (subprocess) with a fixed number of stages. The idea is to use value iteration for the subprocess which have a finite number of stages and a modified version of policy iteration for the main process with infinite planning horizon. Thus only the transition matrix of the main process has to be inverted. Exact solutions can be given to even very extensive problems if it is possible to model the problem in such way that the number of states of the main process is not too large.

The purpose of the present paper is partly to describe the theory of hierarchic Markov processes and the associated optimization algorithm and partly to show how the theory is applied to replacement models. The latter purpose is fulfilled through examples with basis in a stochastic version of the classical constant chain replacement model.

## 2. The policy iteration method

In order to obtain a frame of reference when the theory of hierarchic Markov processes is introduced, the traditional version of a Markov decision process and the policy iteration method of Howard [5] is summarized in this section.

Consider a discrete time Markov decision process with a finite state space  $\Omega = \{1, 2, \dots, \omega\}$  and a finite action set  $D$ . A *policy*  $s$  is a map assigning to each state  $i \in \Omega$  an action  $s(i) \in D$ . Let  $r_i^d$  be the *reward* gained when the state  $i$  is observed, and action  $d \in D$  is taken, and let  $p_{ij}^d$  be the *transition probability* from state  $i$  to state  $j$  if the action  $d$  is taken. As usual,  $p_{ij}^d \geq 0$  for all  $i, j, d$ , and  $\sum_{j=1}^{\omega} p_{ij}^d = 1$  for all  $i, d$ . The time interval between two transitions is called a *stage*. The *discount* factor  $\beta_i^d$  is allowed to depend on the action  $d$  and the present state  $i$ . In practice this means that the stage length  $l_i^d$  may depend on  $d$  and  $i$ . We assume that  $l_i^d > 0$  for all  $i$  and  $d$ , and since  $\beta_i^d$  is defined as  $\beta_i^d = \exp(-\rho l_i^d)$ , where  $\rho$  is the interest rate, we have that  $\beta_i^d < 1$  for all  $i$  and  $d$ . If  $s(i) = d$ , the symbols  $r_i^d$ ,  $p_{ij}^d$ , and  $\beta_i^d$  are also written as  $r_i^s$ ,  $p_{ij}^s$ , and  $\beta_i^s$  respectively. Let  $r^s$  denote the vector  $(r_1^s, \dots, r_{\omega}^s)'$ , and let  $P^s$  be the

matrix  $\{p_{ij}^s\}$ . The diagonal matrix whose non-zero elements are  $\beta_1^s, \dots, \beta_\omega^s$  is denoted  $B^s$ .

If an infinite planning horizon is assumed the vector of *present values*  $g^s = (g_1^s, \dots, g_\omega^s)'$  under the policy  $s$  is calculated as

$$g^s = (I - B^s P^s)^{-1} r^s, \quad (1)$$

where  $I$  is the  $\omega \times \omega$  identity matrix. An *optimal* policy is defined as a policy which maximizes the elements of  $g^s$  (i.e. a policy  $s'$  is optimal if and only if it satisfies the condition  $g^{s'} = \max_s \{g^s\}$ , where max is understood elementwise). The policy iteration method can now be stated as follows:

- (1) Choose an arbitrary policy  $s$ . Go to step (2).
- (2) Calculate  $g^s = (I - B^s P^s)^{-1} r^s$ . Go to step (3).
- (3) For each state  $i$ , determine the action  $d' \in D$  that maximizes  $r_i^{d'} + \beta_i^{d'} \sum_{j=1}^{\omega} p_{ij}^{d'} g_j^s$  and put  $s'(i) = d'$ . If  $s' = s$  then stop since an optimal policy is found. Otherwise redefine  $s$  according to the new policy (i.e. put  $s = s'$ ) and go back to step (2).

Howard [5] showed that the above stated algorithm always provides an optimal policy in a finite number of iteration steps when the discount factor is assumed to be fixed. His proof is, however, easily extended to the situation where the discount factor depends on  $s$  and  $i$ . The latter version was used by Kristensen and Østergaard [10] in a dairy cow replacement model with 177 states and 4 actions. The optimization thus called for inversion of a matrix of the dimension  $177 \times 177$ . Nevertheless a larger number of states was wanted in order to obtain a more accurate representation of a dairy cow, but the transition matrix was already so large that the model could not be extended without making the inversion impossible to carry out. To circumvent this problem the theory of a hierarchic Markov process was developed.

### 3. Hierarchic Markov processes

Consider a discrete time Markov decision process (subprocess) with  $N$  stages and a finite state space  $\Omega_n = \{1, \dots, \omega_n\}$  for stage  $n$ ,  $1 \leq n \leq N$ . The action set  $D_n$  of the  $n$ -th stage is assumed to be finite too. A policy  $s$  of the subprocesses is a map assigning to each stage  $n$  and state  $i \in \Omega_n$  an action  $s(n, i) \in D_n$ . The set of all possible policies

of the process is denoted  $\Delta$ . When the state  $i$  is observed at the stage  $n$  and the action  $s(n, i)$  is taken, a reward  $r_i^s(n)$  is gained. Let  $p_{ij}^s(n)$  be the transition probability from state  $i$  to state  $j$  where  $i$  is the state of the  $n$ -th stage,  $j$  is the state of the following stage, and the policy  $s$  defines the action  $s(n, i)$  to be taken. In this formulation the transition matrix  $P_n^s = \{p_{ij}^s(n)\}$  is not necessarily a square matrix. As before  $p_{ij}^s(n) \geq 0$  for all  $i, j, n$ , and  $s$ , and  $\sum_{j=1}^{\omega_{n+1}} p_{ij}^s(n) = 1$  for all  $i, n$  and  $s$ . The vector  $P_0 = (p_1(0), \dots, p_{\omega_1}(0))'$  gives the probability distribution of the states at stage 1. The stage length  $l_i^s(n)$  is assumed to be determined by the stage, the state and the action taken. In the subprocess the assumption  $l_i^s(n) > 0$  is not needed. The discount factor is defined as  $\beta_i^s(n) = \exp(-\rho l_i^s(n))$  where  $\rho$  is the interest rate. An additional reward  $T$  is gained at the end of stage  $N$ . As before, the parameters may be indexed by actions or policies as convenient.

Let  $f_i^s(n)$  denote the present value of the total expected future reward, excluding  $T$ , under the fixed policy  $s$  for a process in state  $i$  at stage  $n$ . These values can be determined reversely step by step by the well known recurrence equations

$$f_i^s(N) = r_i^s(N) \quad (2)$$

and

$$f_i^s(n) = r_i^s(n) + \beta_i^s(n) \sum_{j=1}^{\omega_{n+1}} p_{ij}^s(n) f_j^s(n+1), \quad (3)$$

where  $n = N-1, \dots, 1$ , and  $i = 1, \dots, \omega_n$ . The present value of the entire process is

$$f^s = \sum_{i=1}^{\omega_1} p_i(0) f_i^s(1). \quad (4)$$

The equations (2)–(4) make it possible to calculate the present value (excluding  $T$ ) of the entire process under any policy.

An optimal policy of the subprocess is defined as a policy  $s'$  satisfying  $g_i^{s'}(n) = \max_{s \in \Delta} \{g_i^s(n)\}$  for all  $i$  and  $n$ , where  $g_i^s(n)$  is the present value including the terminal reward  $T$ . For convenience the present values corresponding to an optimal policy will be denoted  $g_i(n)$ . If  $T$  is known, an optimal policy can be determined by the value iteration method. In each state at each stage, an optimal action is chosen according to the recur-

rence equations

$$g_i(N) = \max_{d \in D_N} \{ r_i^d(N) + \beta_i^d(N)T \} \quad (5)$$

and

$$g_i(n) = \max_{d \in D_n} \left\{ r_i^d(n) + \beta_i^d(n) \sum_{j=1}^{\omega_{n+1}} p_{ij}^d(n) g_j(n+1) \right\}, \quad (6)$$

where  $n = N-1, \dots, 1$ , and  $i = 1, \dots, \omega_n$ .

We shall use a Markov decision process as the one just described as a subprocess of a hierarchic Markov process. Assume that we have a set of  $h$  possible subprocesses. Consider a Markov decision process with an infinite number of stages and the finite state space  $H = \{1, \dots, h\}$ . Each stage in this process, which we shall call the main process, represents a particular subprocess. The action sets of the main process are the sets  $\Delta_i$ ,  $i = 1, \dots, h$ , of all possible policies of the individual subprocesses (to avoid ambiguity the states of the main process will be denoted by Greek letters  $\iota$ ,  $\kappa$  etc). A policy  $S$  is a map assigning to each state  $\iota \in H$  of the main process an action  $S(\iota) \in \Delta_i$ . The transition matrix of the main process is of the dimension  $h \times h$ , and it is denoted  $\Theta = \{\theta_{\iota\kappa}\}$ . The transition probabilities are assumed to be independent of the action taken (in Section 4.4 this assumption is relaxed). The reward in the state  $\iota$  of the main process is the present value  $f_i^s$  (where  $s = S(\iota)$ ) of the entire subprocess calculated as shown in the equations (2)–(4). The index  $\iota$  refers to the state of the main process, and at the same time it points out the corresponding subprocess.

The stage length of the main process in state  $\iota$  is equal to the entire length of the corresponding subprocess. The stage lengths of the subprocesses depend on the observed states, and consequently the stage length  $L$  of the main process is a random variable whose distribution depends on the state  $\iota$  and the policy  $S$ . Since the discount factor  $B$  is defined as  $B = \exp(-\rho L)$  this factor is random too. It is assumed that  $L > 0$  and thus  $0 < B < 1$ . The present value of the infinite main process under the policy  $S$ , when the present state is  $\iota$ , is denoted  $\gamma_i^S$ . By convention, let the symbol  $E^S(\cdot | \iota)$  denote a conditional expectation under the policy  $S$  given that the present state is  $\iota$ . Similarly the

symbol  $E^S(\cdot | \iota, \kappa)$  denotes a conditional expectation given that the state at the present stage is  $\iota$  and the state at the following stage will be  $\kappa$ . Analogous symbols are used for conditional probabilities.

Since the present value of an infinite process under a fixed policy is constant over stages we get for all  $S$

$$\gamma_i^S = f_i^S + E^S(B\gamma_K^S | \iota), \quad \iota = 1, \dots, h, \quad (7)$$

where  $K$  is the (random) state at the following stage. From (7) we have for all  $S$

$$\begin{aligned} \gamma_i^S &= f_i^S + \sum_{\kappa=1}^h E^S(B\gamma_K^S | \iota, \kappa) P^S(K = \kappa | \iota) \\ &= f_i^S + \sum_{\kappa=1}^h E^S(B\gamma_K^S | \iota, \kappa) \theta_{\iota\kappa} \\ &= f_i^S + \sum_{\kappa=1}^h E^S(B | \iota, \kappa) \theta_{\iota\kappa} \gamma_K^S, \quad \iota = 1, \dots, h. \end{aligned} \quad (8)$$

Define the matrix  $\Theta_B^S = \{E^S(B | \iota, \kappa) \theta_{\iota\kappa}\}$  and the vectors  $\gamma^S = (\gamma_1^S, \dots, \gamma_h^S)'$  and  $f^S = (f_1^S, \dots, f_h^S)'$ . The equations (8) can be written in matrix notation as

$$\gamma^S = f^S + \Theta_B^S \gamma^S. \quad (9)$$

The elements of  $\Theta_B^S$  are all non-negative, and the sum of any row is strictly less than 1 in magnitude. Thus it is easily seen that the matrix  $(I - \Theta_B^S)$ , where  $I$  is the  $h \times h$  identity matrix, is non-singular. Thus we can solve equation (9) with respect to  $\gamma^S$ :

$$\gamma^S = (I - \Theta_B^S)^{-1} f^S. \quad (10)$$

Equation (10) is analogous to Equation (1), and Howard's proof [5] of the properties of the policy iteration method is easily extended to cover the present case where an optimal policy is defined as a policy that maximizes the elements of  $\gamma^S$  in equation (10). Then in principle the policy iteration method can be used directly for optimization. However step (3) in Section 2 may seem prohibitive. In the notation used for the main process this point tells us to determine the action  $s \in \Delta_i$  that

maximizes

$$f_t^s + \sum_{\kappa=1}^h E^s(B|\iota, \kappa) \theta_{\iota\kappa} \gamma_{\kappa}^S, \quad (11)$$

where  $S$  is the policy determined during the previous step of the iteration cycle. Since the action  $s$  is an entire policy of a subprocess the number of alternatives can be extremely large (although finite). The exact number is  $\prod_{n=1}^N (m_n)^{\omega_n}$ , where  $m_n$  is the number of elements in the action set  $D_n$  of a subprocess. It is obvious that even relatively small models contain a prohibitive number of policies.

It is, however, not necessary to try all alternatives. The value iteration method as stated in the recurrence equations (5) and (6) gives directly the policy that maximizes the expression (11). We only have to put  $T = T_t = \sum_{\kappa=1}^h \theta_{\iota\kappa} \gamma_{\kappa}^S$  and use the equation (5) and (6) to determine the desired policy of the subprocess. Intuitively this is obvious, and since the formal proof is easily accomplished it is omitted. With this modification of policy iteration, the optimization method of hierarchic Markov processes can be stated as follows:

- (1) Choose an arbitrary policy  $S$  of the main process. Go to step (2).
- (2) Calculate  $\gamma^S = (1 - \Theta_B^S)^{-1} f^S$ . Go to step (3).
- (3) For each state  $\iota$  of the main process, determine by use of the recurrence equations (5) and (6) the optimal policy  $s' \in \Delta_{\iota}$  of the  $\iota$ -th subprocess when  $T = T_t = \sum_{\kappa=1}^h \theta_{\iota\kappa} \gamma_{\kappa}^S$ , and put  $S'(\iota) = s'$ . If  $S' = S$  then stop since an optimal policy is found. Otherwise redefine  $S$  according to the new policy (i.e. put  $S = S'$ ) and go back to step (2).

The calculation of the conditional expectation  $E^S(B|\iota, \kappa)$  depends on the individual model formulation. A method for an important class of replacement models is shown in Section 4.5.

#### 4. Hierarchic Markov processes and replacement models

##### 4.1. Background

In a review article on Markov decision processes van der Wal and Wessels [24] concluded that the

theoretical aspects are now fairly well elucidated, but that the most important research area for the years to come is the development of methods for handling very large problems since most real problems give rise to a tremendous number of states. One method, called truncated policy iteration, suggested that instead of solving equation (1) directly by inversion of  $(I - B^s P^s)$  the solution is approximated by use of a truncated series based on the relation  $(I - B^s P^s)^{-1} = \sum_{n=0}^{\infty} (B^s P^s)^n$ . The rate of convergence of such methods is discussed by Dembo and Haviv [3]. In a dairy cow replacement study, Ben-Ari and Gal [2] used a method called parameter iteration to approximate an optimal solution in a very large model with 180 state variables. The idea was to combine simulation and dynamic programming to compute successive linear approximations.

Also the present paper is a contribution to the efforts concerning the practical analysis of large problems, but instead of using some operational methods to approximate an optimal solution, the special structure of a replacement model is used to introduce a hierarchic model where the dimension of the matrix to be inverted in equation (10) is small even though the model may be very large. In that way we are able to obtain an exact solution.

One of the reasons that a replacement model, formulated as a traditional Markov decision process, is usually very large is that the age of the asset in question is included as a state variable. Consider a replacement model with a finite state space  $\Omega = \{1, \dots, \omega\}$ . Assume that one of the state variables defining  $\Omega$  is the age of the asset denoted as  $a$ . In a discrete time model only a finite number of values are distinguished so that  $a \in \{0, 1, \dots, A\}$ . If the value of  $a$  increases one unit for each stage it is obvious that a transition from state  $i$ , where  $a = a_i$ , to state  $j$ , where  $a = a_j$ , is only possible in one step if  $a_j = a_i + 1$ . Thus we have for any policy that  $p_{ij}^s = 0$  if  $a_j \neq a_i + 1$ , and  $p_{ij}^s \geq 0$  if  $a_j = a_i + 1$ .

If several other state variables are included in the model the transition matrix  $P^s = \{p_{ij}^s\}$  may be very large, but because of the variable  $a$  concerning age, the main part of the elements of  $P^s$  are zero. Thus the only information, they carry, is that such transitions are not possible. Because of the large transition matrix, the policy iteration method becomes difficult to implement if more than a few state variables are included.

The formulation of hierarchic Markov processes circumvents this problem concerning age as a state variable. The age of the asset does not need to be included in any state space. Instead the age is accounted for by the stage number of the subprocess as shown in the following section. The idea of a hierarchic Markov process is to take advantage of the fact that when a replacement takes place something fundamentally different begins in the process. In the traditional Markov decision model a replacement is only represented as a transition just like all others from one state to another.

#### 4.2. Only one state in the state space of the main process

Consider an asset which is described by the age  $a \in \{0, 1, \dots, A\}$  and some state variables  $x_1, \dots, x_v$ . Define a subprocess with  $N = A + 2$  stages so that the length of the entire subprocess corresponds to the life time of the asset. In other words, as long as the asset is not replaced, the age of the asset is measured by the stage number. The state spaces of the process are defined by the state variables  $x_1, \dots, x_v$  which are assumed to be discrete. Define further in each state space a replacement state referred to as  $\lambda$ . For  $n = 1, \dots, N - 2$  the action set is  $D_n = \{\text{'keep'}, \text{'replace'}\}$ , but for  $n = N - 1$ , where the age is  $A$ , only one action is allowed:  $D_{N-1} = \{\text{'replace'}\}$ . No action is taken for  $n = N$  (i.e.  $D_N = \emptyset$ ). If  $s(n, i) = \text{'keep'}$  the reward  $r_i^s(n)$  is equal to the net revenue of the asset during the  $n$ -th stage when state  $i$  is observed. If  $s(n, i) = \text{'replace'}$  the asset is replaced immediately, and the reward  $r_i^s(n)$  is equal to the salvage value of the asset at the age indicated by  $n$  when the present state is  $i$ . In that case the stage length is zero (i.e.  $\beta_i^s(n) = 1$ ), and the process moves on to state  $\lambda$ . Thus we have

$$p_{i\lambda}^s(n) = 1 \quad \text{if } s(n, i) = \text{'replace'}. \quad (12)$$

For state  $\lambda$  we further assume for all  $s$  that

$$p_{\lambda\lambda}^s(n) = 1, \quad n = 1, \dots, N - 1, \quad (13)$$

$$r_{\lambda}^s(n) = 0, \quad n = 1, \dots, N, \quad (14)$$

and

$$\beta_{\lambda}^s(n) = 1, \quad n = 1, \dots, N. \quad (15)$$

In other words, the stages from the replacement to the end of the subprocess are fictive without any economic consequences. Since the action taken

at stage  $N - 1$  is always 'replace', the state space at stage  $N$  only contains one element:  $\Omega_N = \{\lambda\}$ .

All stages, where the present state is not  $\lambda$  and the action taken is 'keep', are assumed to be of equal length so that we have  $\beta_i^s(n) = \beta < 1$ , where  $\beta$  is fixed for  $s(n, i) = \text{'keep'}$ , and  $i \neq \lambda$ .

When the action 'replace' is taken, the asset is replaced by a new asset which is assumed to be identical to the present one. Thus this model is a stochastic version of the constant chain approach as it was introduced in [12] (cf. Section 1). A stage in the main process begins when a new asset is purchased, and it ends when it is replaced. The state space of the main process only contains one state (i.e.  $H = \{1\}$ ). Then the transition matrix  $\Theta$  only contains the element 1, and the inversion in equation (10) is just the inversion of a real number.

Even in this rather degenerate hierarchic Markov process with only one state in the main process there is a large computational advantage of using the hierarchic structure, because we only have to invert a real number in equation (10). If the age was included as a state variable in the subprocess the resulting model would be an ordinary Markov decision process as applied in for instance [10]. Such model might very well contain several thousand states, and then we had to invert a matrix of that dimension if traditional policy iteration was used.

#### 4.3. Several states in the state space of the main process

Usually it is convenient to use the possibility of having several states in the main process. Suppose that the assets in question can be sorted by quality in  $h$  different classes  $i = 1, \dots, h$ . When a new asset is ordered, it is not possible to specify the desired quality, but when it is delivered, the quality class  $i$  can be determined by some test. Then there will be  $h$  different subprocesses, and the state of the main process is determined by the quality of the asset. If the probability of an asset being of the quality  $i$  is denoted  $q_i$ ,  $\sum_{i=1}^h q_i = 1$ , we have for all  $i$

$$\theta_{i\kappa} = q_{i\kappa}. \quad (16)$$

In this formulation even very extensive models can be optimized. In the dairy cow replacement model of Kristensen [7] the state variables of the

subprocesses were the length of the calving interval (3–8 classes), the milk yield of the present lactation (15 classes), and at some stages the milk yield of the previous lactation (15 classes). The number of stages was 25. The state variable of the main process was the genetic class (5 classes) defined from the father of the cow. If that model were to be formulated in the traditional manner as a Markov decision process the genetic class, the age (24 classes), and the state variables of the subprocess should be included as state variables. In that case the state space would contain more than 60 000 states, and a matrix of corresponding dimension should be inverted in equation (1). By using a hierarchic Markov process it was nevertheless possible to give an exact solution, because only a matrix of the dimension  $5 \times 5$  should be inverted in equation (10). A detailed description of this model is given in [9], and the estimation of parameters is described by Kristensen [8].

#### 4.4. An extended model

A perhaps more realistic way of using the possibility of having several states in the main process involves a slight change in the formulation of the hierarchic Markov process. The assumption of the transition matrix  $\Theta$  being independent of the policy of the main process is relaxed. Assume again that an asset exists in  $h$  different qualities, but that we are able to rank the qualities from the least preferred to the most preferred. The state number ( $\iota = 1, \dots, h$ ) is set equal to the rank of the corresponding quality so that  $\iota_2$  is preferred to  $\iota_1$  if and only if  $\iota_2 > \iota_1$ . Unlike the former example we now assume that we can order a specific quality, but that there is a limited supply so that if we order the quality  $\iota$ , there is the probability  $\pi_\iota$  that it can be delivered. The only restriction on the probabilities  $\pi_1, \dots, \pi_h$  is the usual  $0 \leq \pi_\iota \leq 1$ , for  $\iota = 1, \dots, h$ . The relation of  $\pi_1, \dots, \pi_h$  to the transition probabilities appear below from equation (19).

In the subprocesses we define a number of new actions of the kind ‘replace if a new asset of at least quality  $\iota$  is available’. The action stated is referred to as action  $\iota$ . Further we refer to the action ‘keep’ as action  $h+1$  so that the total number of actions is  $h+1$ . Instead of the single replacement state  $\lambda$  we now assume that all state spaces contain  $h$  replacement states  $\lambda_1, \dots, \lambda_h$ .

When a subprocess ends up in state  $\lambda_\iota$  it means that the old asset is replaced by a new asset of the quality  $\iota$ , and that the next state of the main process will be  $\iota$ . At the end of a subprocess, the state will always be one of the replacement states  $\lambda_1, \dots, \lambda_h$  (i.e.  $\Omega_N = \{\lambda_1, \dots, \lambda_h\}$ ). A change in the formulation concerns equation (5) where the terminal reward  $T$  now depends on the state  $i \in \Omega_N$ . The equation should now be states as

$$g_i(N) = \max_{d \in D_N} \{r_i^d(N) + \beta_i^d(N)T_i\}, \quad (17)$$

where

$$T_i = \gamma_i^S, \quad i \in \Omega_N. \quad (18)$$

If the action  $h+1$  (‘keep’) is taken, the rewards and the transition probabilities are the same as when the action was ‘keep’ in the preceding sections, but if the action  $\iota$ ,  $\iota \in \{1, \dots, h\}$  is taken the probability of a transition to state  $\lambda_\kappa$  is

$$p_{i\lambda_\kappa}^\iota(n) = \begin{cases} 0, & \kappa < \iota, \\ \pi_\kappa \prod_{j=\kappa+1}^h (1 - \pi_j), & h > \kappa \geq \iota, \\ \pi_\kappa, & \kappa = h, \end{cases} \quad (19)$$

$$n = 1, \dots, N-1.$$

The probability that the asset is kept is  $1 - p_{i\lambda_\kappa}^\iota(n)$ . All replacement states are assumed to possess the properties stated in the equations (13)–(15) and further  $p_{\lambda_\iota\lambda_\kappa}^s(n) = 0$  if  $\iota \neq \kappa$ .

This reformulation also concerns the transition probabilities of the main process. The actions of the subprocess influence the following state of the main process, so that the transition probabilities  $\theta_{i\kappa}$  depend on the policy  $s$  and should really be denoted as  $\theta_{i\kappa}^s$  or  $\theta_{i\kappa}^S$ . In equation (10) these probabilities are needed. They are calculated as follows. The subprocess  $\iota$  will ultimately end up in one of the states  $\lambda_1, \dots, \lambda_h$ . At each stage  $n$  of the subprocess we can calculate the probability distribution  $\psi^{\iota ns}$  of the states in  $\Omega_n$  under the policy  $s$  from the recurrence equations

$$\psi^{\iota,1,s} = P_0 \quad (20)$$

and

$$\psi^{\iota ns} = \psi^{\iota, n-1, s} P_{n-1}^s, \quad n = 2, \dots, N. \quad (21)$$

The transition probabilities  $\theta_{i\kappa}^s$  are then given as

$$\theta_{i\kappa}^s = \psi_{\kappa}^{\iota N s}, \quad (22)$$

where  $\psi_{\kappa}^{\iota N s}$  (the  $\kappa$ -th element of  $\psi^{\iota N s}$ ) denotes the probability that the subprocess  $\iota$  is in state  $\lambda_{\kappa}$  at stage  $N$ .

The algorithm stated in Section 3 is also valid in this case. The only change is that equation (5) is replaced by equation (17) and (18) and that the transition probabilities of the main process now depend on the policy.

#### 4.5. The calculation of the conditional expectation $E^s(B | \iota, \kappa)$

In the examples given, the conditional expectation  $E^s(B | \iota, \kappa) = E^s(B | \iota, \kappa)$ , where  $S(\iota) = s \in \Delta_{\iota}$ , can be calculated quite easily. We have for all  $\iota, \kappa, s$

$$E^s(B | \iota, \kappa) = \sum_{n=1}^N E^s(B | \iota, \kappa, \eta = n) \times P^s(\eta = n | \iota, \kappa) \quad (23)$$

where  $\eta$  denotes the stage of the subprocess  $\iota$  where the replacement takes place, and  $P^s(\eta = n | \iota, \kappa)$  denotes the conditional probability under the policy  $s$  that the asset is replaced at stage  $n$  given that the present subprocess is  $\iota$ , and the following subprocess will be  $\kappa$ .

In the example of Section 4.2, equation (23) reduces to

$$E^s(B) = \sum_{n=1}^N E^s(B | \eta = n) P^s(\eta = n), \quad (24)$$

since there is only one state in the state space  $H$  of the main process. In this situation we have for all  $n$  and  $s$

$$E^s(B | \eta = n) = \beta^n, \quad (25)$$

and

$$P^s(\eta = n) = \psi_{\lambda}^{1, n, s} - \psi_{\lambda}^{1, n-1, s} \quad (26)$$

where the probability distributions  $\psi^{1, n-1, s}$  and  $\psi^{1, n, s}$  are calculated from the recurrence equations (20) and (21) with  $\iota = 1$ .

In the example of Section 4.3 the states of different stages are independent (cf. equation (16)) so equation (23) can for all  $\iota$  and  $s$  be reduced to

$$E^s(B | \iota) = \sum_{n=1}^N E^s(B | \iota, \eta = n) P^s(\eta = n | \iota). \quad (27)$$

Analogous to equation (25) we have for all  $\iota, n$ , and  $s$

$$E^s(B | \iota, \eta = n) = \beta^n, \quad (28)$$

and

$$P^s(\eta = n | \iota) = \psi_{\lambda}^{\iota, n, s} - \psi_{\lambda}^{\iota, n-1, s}. \quad (29)$$

Also in the example of Section 4.4 we have for all  $\iota, \kappa, n$ , and  $s$

$$E^s(B | \iota, \kappa, \eta = n) = \beta^n. \quad (30)$$

Since there are  $h$  replacement states  $\lambda_1, \dots, \lambda_h$  and the successive states of the main process are no longer independent the calculation of  $P^s(\eta = n | \iota, \kappa)$  is more complicated. We get for all  $\iota, \kappa, n$ , and  $s$

$$P^s(\eta = n | \iota, \kappa) = (\psi_{\lambda_{\kappa}}^{\iota, n, s} - \psi_{\lambda_{\kappa}}^{\iota, n-1, s}) / \theta_{\iota \kappa}^s. \quad (31)$$

## 5. Conclusions

The introduction of the concept of a hierarchic Markov process can be considered as a contribution to the theory of Markov decision processes which makes it possible to analyse even very large problems with a certain structure. Practical experience has demonstrated that the optimization method, which is a modified version of the policy iteration method, provides exact solutions even to problems which would contain more than 60 000 states if they were to be formulated as ordinary Markov decision processes. The idea is to define the hierarchic process in such a way that the state space of the main process is small even though the state spaces of the subprocesses may be very large.

The method has been developed to take advantage of the special structure of a replacement problem. Through a number of examples it is shown how the theory is applied in replacement models based on a stochastic version of the constant chain approach. It is demonstrated how the replacement decisions can depend on the quality of the asset available for replacement.

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# Optimal Replacement and Ranking of Dairy Cows Determined by a Hierarchic Markov Process

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## ABSTRACT

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A dairy cow replacement model based on the notion of a hierarchic Markov process is presented. A hierarchic Markov process is a series of Markov decision processes built together in one Markov decision process, called the main process.

In the model a cow is described in terms of lactation number, stage of lactation, the level of milk yield during the previous and present lactation, the length of the calving interval, and the genetic class defined from the breeding value of the father. The criterion of optimality is the maximization of the present value under an infinite planning horizon. Revenues from milk, calves, and replaced cows, feed costs and costs of replacement heifers are considered.

The future profitability calculated from the optimal solution is used for ranking of the cows in the herd. The genetic class makes it possible to include the heifers available for replacement in the ranking and to let the replacement decision depend on the genetic class of the heifers.

It is concluded that the milk yield of previous lactation is not needed as a state variable when the other variables are present in the model.

## INTRODUCTION

In traditional replacement theory most attention has been paid to industrial items. Typically the optimal replacement time has been determined by considerations based on comparison of the marginal net revenue of the old item and the average net return of the new item. The dairy cow replacement problem is different in three main features (Ben-Ari et al., 1983): firstly, the stochastic element is prevalent; secondly, there is only a limited supply of replacement heifers available (typically home-grown) and lastly the production of the dairy cow is cyclic.

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These features make the marginal net revenue approach less suitable for the case of dairy cows, and dynamic programming (in particular the Markov decision process) has become widely accepted as the main tool for optimization. A review of studies in this field is given by van Arendonk (1984). Recently papers have been presented by van Arendonk (1985a, 1986) and van Arendonk and Dijkhuizen (1985) where a comprehensive work of identifying the most important traits and conditions in the determination of optimal replacement policies is carried out. In van Arendonk (1985b) the conclusions from the former papers are used for a discussion of management guidelines to support replacement and insemination decisions concerning individual cows.

Several optimization methods are available to Markov decision processes. From a theoretical point of view, the policy iteration method (Howard, 1960) should be preferred since it is exact and very efficient in the sense that convergence occurs rapidly (van der Wal and Wessels, 1985). A problem however, is the size of the state space since the method implies that a transition matrix of the dimension  $n \times n$  (where  $n$  is the number of states) is inverted. The value iteration method, which has been used in most studies, makes it possible to handle models with a very large number of states and allows continuous genetic improvement to be taken into account. Policy iteration does not directly allow this, but the phenomenon can be simulated indirectly by adjustment of the interest rate used in the optimization. On the other hand, the value iteration method suffers from the weakness that it is not exact when an infinite planning horizon is used. Further, the convergence is often rather slow.

In an unpublished dairy cow replacement study, Kristensen (1985) introduced an alternative formulation of a Markov decision process called a hierarchic Markov process, where each stage itself represents a Markov decision process. The biological part of the study, describing a model of milk yield (including estimation of the parameters) accounting for the effect of the herd level, the breeding value of the father, and the length of the calving interval is published in Kristensen (1986a). In that paper the effect of milk yield during previous periods on present milk yield as well as probability distributions of the length of the calving interval are also estimated. The formulation of the theory of hierarchic Markov processes is presented in Kristensen (1986b).

The aim of the present paper is to present a dairy cow replacement model based on a hierarchic Markov process, and to present some of the results when the model is used in the determination of an optimal replacement policy under conditions referring to a typical price and interest situation of a Danish dairy herd in 1985.

## THE REPLACEMENT MODEL

### *Hierarchic Markov processes*

The idea of a hierarchic Markov process is that each stage in the main process represents a separate Markov decision process (a subprocess) with a finite

number of stages. The number of alternative subprocesses is equal to the number of states in the main process. If state  $i$  is observed in the main process it means that the  $i$ th subprocess is running at present. State variables are defined in the main process as well as in the subprocesses. Actions (decisions) are only defined in the subprocesses because an action in the main process is equal to an entire policy of the subprocess in question. The rewards (net revenues of a single stage) in the main process are calculated from the rewards of the subprocess, and the stage length in the main process is equal to the total length of the corresponding subprocess. For a complete description, reference is made to Kristensen (1986b). The criterion of optimality is the maximization of the expected present value of the total future rewards under infinite planning horizon.

### *The subprocesses*

A stage in a subprocess ends and the next one begins 16, 24, 32 and 40 weeks after calving and always immediately when a cow is replaced. The length of the stages beginning 40 weeks after calving depends on the length of the calving interval. Other stages are 8 weeks long if the cow is kept, but if the cow is replaced at the beginning of a stage, the length is zero.

The states are defined by the values of three state variables which are the milk yield (fat corrected milk) of the previous lactation (15 classes), the milk yield of the present lactation (15 classes), and the calving interval (a maximum of 8 classes: 44, 48, 52, ..., 72 weeks). A measure of milk yield corrected for the herd level, the breeding value of the father, and the calving interval as described in Kristensen (1986a) is used. The classes of milk yield are defined in such a way that all 15 classes occur with the same probability in a herd which has not been subjected to culling. Because of too few observations in the fifth and sixth lactations it was not possible to estimate the effect of the milk yield of the previous lactation on future milk yield (Kristensen, 1986a). In those cases (later than 24 weeks after the fifth calving) the milk yield of the previous lactation was omitted as a state variable.

The number of classes corresponding to the calving interval varies with the stage number as it appears in Table I, where the characteristics of the state variables are summarized. At 16 weeks after calving, for instance, only three classes are distinguished: '44 weeks', '48 weeks', and '52 weeks or more'.

Beyond the states defined by the state variables mentioned, three additional states are defined. When a cow is replaced the subprocess immediately transfers to the 'replacement state' where it will stay during the remaining stages. The stage length in this state is always zero, and the reward is also zero. Such stages are fictitious, and they only serve as a way of ensuring that all subprocesses contain the same number of stages.

Another state is the 'disease state'. When this state is observed it means that

TABLE I

The state variables and the number of classes distinguished at various stages

Lactation number	Lactation stage	State variable		
		Milk yield of previous lactation	Milk yield of present lactation	Length of the calving interval
1	16	—	15	3
1	24	—	15	5
1	32	—	15	7
1	40	—	15	8
2-4	16	15	15	3
2-4	24	15	15	5
2-4	32	15	15	7
2-4	40	15	15	8
5	16	15	15	3
5	24	15	15	5
5	32	—	15	7
5	40	—	15	8
6	16	—	15	3
6	24	—	15	5
6	32	—	15	7
6	40	—	—	—

the cow is in such a condition that it must be replaced (involuntarily) irrespective of the values of the usual state variables observed. The cow is replaced at once (i.e. the stage length is zero) and the process transfers to the replacement state.

If the cow is not known to be pregnant 40 weeks after calving the process occupies the 'infertility state'. In that case the cow is replaced immediately with the same consequence to the subprocess as in the disease state.

The weight of the cow is not included as an ordinary state variable. Instead a standard weight curve (Kristensen, 1986a) describing weight as a function of lactation number and stage of lactation is used.

It is assumed that the maximum age of a dairy cow is reached 40 weeks after the sixth calving; this is the latest time at which the animal is replaced. This maximum age is based on the fact that a typical annual replacement rate in Danish dairy herds is between 40% and 50% so that only very few cows reach the sixth lactation.

The actions of the subprocesses are (1) replace the cow with a heifer (immediately); (2) keep the cow.

The rewards are calculated separately for each combination of subprocess, stage, state and action. The reward, when the cow is kept, is based on revenues from milk and calves (if a calving takes place during the stage) and feeding costs (basic food and food for milk production). At the first stage of the subprocess the price of the heifer is deducted. If the cow is replaced the reward is equal to the salvage value of the cow.

### *The main process*

A stage in the main process ends and a new one begins when a replacement takes place. The length of the stage is thus equal to the herdlife of the individual cow. The only state variable included in the main process is the breeding value of the father of the cow, calculated with respect to the milk yield of his offspring. Five classes are distinguished, and the variable will be referred to as the genetic class. The definition of the various classes from relative breeding values is as follows: Class 1:  $\leq 97$ , Class 2: 98–102, Class 3: 103–107, Class 4: 108–112 and Class 5:  $\geq 113$ . The effect of the genetic class on milk yield was only significant during the first three lactations because of the relatively few observations concerning older cows; this does not mean that there is no effect during later lactations, but that the particular data were too sparse to estimate it. As a consequence, the effect is omitted during the last three lactations.

### *The parameters*

The transition probabilities of the subprocess were calculated from the model of milk yield described by Kristensen (1986a), and the estimated probability distributions of the calving intervals from the same paper. At each stage there is a probability of involuntary replacement independent of the milk production and the calving interval. These probabilities are based on data from Dijkhuizen (1980), and are calculated using a method described by van Arendonk (1985c). The probability of involuntary replacement is equal to the probability of a transition to the disease state. The rewards during stages where the cow is kept are determined from the production and food intake of the cow calculated from the model described by Kristensen (1986a). The prices and the interest rate can readily be changed so that they fit the conditions of the individual herd. If a cow is replaced the salvage value is determined by the live weight of a cow at the lactation and stage of lactation in question. The standard weight curve used appears in Kristensen (1986a).

The transition probabilities of the main process are assumed to be independent of the present state (i.e. the probability of observing a specific state at the next stage is the same no matter which state is observed at the present stage). In practice this is equal to assuming that the genetic class of a heifer is independent of that of the present cow. Naturally such an assumption is not

correct in every detail. If the present cow is from a high genetic class it produces better offspring, and then there is a slightly better material available for later replacements, because there is a small chance that the cow is replaced by its own daughter. In a large herd, however, this chance is very small, so the effect will be negligible. Since the model is most likely to be used in large herds the assumption of independence seems reasonable. Thus only a probability distribution of the genetic class is needed. The five classes were defined in such a way (cf. Kristensen, 1986a) that all classes are observed with approximately the same probability.

### *Discussion of the model*

If the model described in the previous sections were to be formulated as a traditional Markov decision process two additional state variables should be included beyond the three variables of the subprocesses and the genetic class of the main process. One of these additional variables should be the lactation number with six classes, and the other should be the stage of lactation with four classes. The total number of states in such a model would exceed 60 000. It is clear that optimization by use of policy iteration would be out of question.

Other models containing a large number of states have usually been simplified with respect to transitions between different classes of milk yield. Smith (1971) predicted the class of milk yield of the following lactation in a deterministic manner in a model with 15 138 states. Stewart et al. (1977, 1978) assumed that a cow remained in the same class of milk yield during her entire life even though their model only contained 2695 states. In the models of van Arendonk (1985a, 1986) and van Arendonk and Dijkhuizen (1985) it was assumed that transitions between different classes of milk yield only took place at the end of the lactation period. Thus the class remained the same during the entire lactation period. In the present study, however, no such simplifications are made; transitions between different classes of milk yield are possible at any stage of the subprocesses according to the probability distributions defined by the transition probabilities.

The idea of representing the production of the cow by the milk yield of the previous lactation and the milk yield of the present lactation was originally introduced by Smith (1971). Recently van Arendonk (1985a) and van Arendonk and Dijkhuizen (1985) have used the same approach. Several models (beginning with Giaever, 1966) have used the calving interval as a state variable, but until the present study only van Arendonk and Dijkhuizen (1985) and van Arendonk (1986) have defined classes with intervals of a single month. The possibility of taking actions more than once during a lactation was introduced by Kristensen and Østergaard (1982), and it was also used in the models of van Arendonk (1985a, 1986) and van Arendonk and Dijkhuizen (1985).

TABLE II

Standard conditions used in the optimization

Prices (Dkr.)	
Milk (kg FCM)	2.40
Basic food (SFU <sup>1</sup> )	1.30
Food for milk production (SFU)	1.90
Calf	1400.00
Heifer	8500.00
Young cow until 2nd calving (kg. live weight)	11.50
Older cow (kg. live weight)	11.00
Interest rate <sup>2</sup> (%)	3
Herd level <sup>3</sup>	5400

<sup>1</sup>Scandinavian Feed Units.<sup>2</sup>Corrected for inflation.<sup>3</sup>Average milk yield of the first 40 weeks of a lactation in the herd. Adjusted to 1st lactation level (cf. Kristensen, 1986a).

No previous models known to the author have contained a state variable concerning the genetic class. As stated by Ben-Ari et al. (1983) heifers cannot be considered as standard replacement items, as is usually assumed in the traditional replacement theory. The genetic class makes it possible to compare the heifers available in the herd with the present cows and to rank the heifers on sire breeding values.

Genetic improvement is not directly considered, but it must be assumed that the phenomenon can be simulated rather precisely by assuming that a continuous genetic improvement will result in a fixed annual rate of increasing net revenue. If this rate is for instance 1% and the interest rate is say 5% we only have to use an interest rate of 4% to account for genetic improvement in the optimization.

### *Optimization*

The computer programs for practical optimization were written in Pascal. They were constructed in such a way that all parameters, assumptions, and prices can readily be changed without affecting the rest of the programs. This approach makes it possible to optimize under the very conditions of the individual dairy herd. The standard conditions used in the present optimization appear in Table II.

Comprehensive sensitivity analyses were carried out by Kristensen and Østergaard (1982) as well as by van Arendonk (1985a, 1986) and van Arendonk and Dijkhuizen (1985). Because the conclusions of these studies agree in almost every detail no such analyses were included in the present study. Instead attention has been paid to the interpretation of the optimal replace-

ment policy and to the discussion of further applications of the results from the optimization.

## RESULTS AND DISCUSSION

### *Optimal policy and ranking of cows*

Since the optimal policy consists of more than 60 000 actions it is impossible to show it as a whole in a table. Instead examples are given concerning cows at 24 weeks after the fourth calving. Optimal actions for calving intervals of 44 and 56 weeks respectively appear from Table III. The figures of the table give the expected gain of taking the action 'keep' compared to 'replace'; a negative number means that it is optimal to replace in that state. At this lactation number there is no distinction between the five subprocesses (i.e. the genetic class). If optimal actions were shown for cows during the three first lactations we also had to tell which genetic class they belonged to.

The expected gain, which we shall denote as the future profitability (cf. van Arendonk, 1984), can be used to rank the cows of a particular herd. The state and the stage of each cow are determined, and the associated future profitability is calculated, as has been done in Table III for some states concerning the fourth lactation. The size of the future profitability then determines the rank of the cow compared to the other cows in the herd. This ranking is even more important than the optimal policy. Sensitivity analyses in Kristensen and Østergaard (1982) showed that the ranking is almost unchanged even when the prices and the interest rate are varied considerably. The optimal policy is far more sensitive to such changes. In many situations it is also more relevant to the dairy farmer to know which cow is least profitable rather than whether a particular cow should be replaced. Replacements are often dictated by the calvings of new heifers, and then naturally the least profitable cows should be replaced. Such situations often arise in herds that only use home-grown heifers, which is a very common policy in Denmark because of infection risks. The ranking also makes it possible to account for the limited supply of heifers mentioned by Ben-Ari et al. (1983). At any time the ranking selects the least profitable cow, and if a heifer is available this cow is replaced.

### *The state variables of the subprocesses*

The influence of all three state variables is clearly seen in Table III: low yielding cows are replaced, but if the calving interval is short a lower milk yield is accepted before the cow is replaced. The significance of each state variable can be revealed by comparing the future profitability of different states. As expected, the most important state variable is the class of milk yield of the present lactation. The difference in future profitability between the lowest and

TABLE III

Expected gain (Dkr.) if the cow is kept (24 weeks after the 4th calving)<sup>1</sup>

Length of calving interval	Class of milk yield, previous lactation	Class of milk yield, present lactation														
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
44	1	-513	-317	-115	100	309	479	652	813	974	1150	1320	1530	1765	2078	2785
44	2	-471	-269	-55	165	374	543	717	877	1039	1215	1384	1594	1830	2144	2837
44	3	-450	-245	-26	196	406	575	747	909	1071	1246	1416	1626	1862	2176	2862
44	4	-435	-227	-4	220	430	599	771	933	1095	1270	1439	1650	1886	2201	2880
44	5	-421	-210	17	241	451	620	793	954	1116	1291	1461	1671	1907	2223	2896
44	6	-410	-197	33	258	468	637	810	971	1133	1309	1478	1688	1924	2240	2908
44	7	-398	-182	49	275	486	654	828	989	1151	1326	1496	1706	1943	2258	2922
44	8	-388	-169	65	292	502	671	844	1005	1168	1342	1512	1723	1959	2275	2934
44	9	-377	-156	81	309	519	687	860	1022	1184	1359	1528	1739	1975	2292	2945
44	10	-365	-142	99	327	536	704	878	1040	1201	1377	1547	1757	1993	2312	2958
44	11	-354	-129	115	344	554	722	896	1057	1219	1394	1564	1774	2010	2330	2970
44	12	-340	-111	135	365	575	743	917	1078	1240	1415	1585	1795	2032	2352	2985
44	13	-325	-92	158	389	599	766	941	1102	1265	1439	1609	1819	2056	2378	3002
44	14	-304	-65	189	420	630	798	973	1134	1295	1471	1640	1851	2087	2412	3024
44	15	-262	-11	252	485	694	862	1037	1199	1360	1535	1705	1915	2152	2482	3068
56	1	-517	-341	-256	-192	-134	-87	-39	6	50	98	149	227	365	647	1420
56	2	-474	-299	-214	-150	-92	-45	3	48	92	141	192	275	423	714	1473
56	3	-454	-279	-194	-129	-71	-25	24	68	113	162	214	299	451	747	1498
56	4	-439	-263	-178	-114	-56	-9	39	84	128	177	230	317	473	772	1517
56	5	-425	-249	-164	-100	-42	5	53	97	142	191	245	334	492	794	1533
56	6	-413	-238	-153	-88	-31	16	64	109	153	203	256	347	507	812	1546
56	7	-402	-227	-142	-77	-19	28	76	120	165	214	268	361	525	831	1560
56	8	-391	-216	-131	-66	-9	38	86	131	176	225	280	373	540	848	1572
56	9	-381	-205	-120	-56	2	49	97	141	186	236	291	387	555	866	1584
56	10	-369	-194	-109	-44	14	61	109	153	198	248	304	401	571	886	1596
56	11	-358	-182	-98	-33	25	72	120	164	209	259	316	414	587	905	1609
56	12	-344	-169	-84	-19	39	86	134	178	223	273	331	431	608	928	1624
56	13	-328	-153	-68	-3	54	101	149	194	239	289	347	451	631	954	1641
56	14	-308	-133	-48	17	75	122	170	214	259	310	370	476	660	990	1663
56	15	-266	-91	-6	59	117	164	212	256	301	353	416	530	723	1064	1708

<sup>1</sup>Negative numbers refer to states where the optimal action is to replace, and positive numbers refer to states where the optimal action is to keep.

the highest class is approximately 3300 Dkr. for cows with a calving interval of 44 weeks, and approximately 1950 Dkr. for cows with a calving interval of 56 weeks.

The influence of the class of milk yield of the previous lactation is much smaller. The difference in future profitability between the lowest and the highest class is in the range 250–420 Dkr., depending on the calving interval and the milk yield of the present lactation. At earlier stages the difference is even smaller, and at many stages (not shown) the optimal decision does not depend on the milk yield of the previous lactation at all. The difference is a little larger in Table III, probably because the genetic class is not included in the model from the fourth lactation, and then the milk yield of the previous lactation becomes more important in the prediction of future milk yield.

The influence of the calving interval is revealed in the same way. The difference in future profitability between calving intervals of 44 and 56 weeks varies from approximately zero for low yielding cows to approximately 1400 Dkr. for high yielding cows. The reason that there is almost no difference for low yielding cows is that the future profitability is defined as the gain if the cow is kept until the beginning of the next stage where the optimal action of that stage is taken compared to a situation where the cow is replaced immediately. A very low yielding cow, which is kept at the present stage, would almost certainly be replaced at the next stage, irrespective of the calving interval, and future profitability will not depend on the calving interval. On the other hand the influence is considerable for high yielding cows which will probably be kept for several stages. The difference of approximately 1400 Dkr. is equal to the economic advantage of shortening the calving interval from 56 weeks to 44 weeks for a high yielding dairy cow.

### *The influence of genetic class*

The influence of the genetic class can be illustrated using the survival curves of different classes. At each stage the survival curve shows the probability that a heifer which entered the herd at first calving is still present in the herd. In Fig. 1 the survival curve of a heifer from the first genetic class (the lowest class) is compared to that of a heifer from the fifth class (the highest class) and shows the considerable difference in average herd life of the two classes. This is perhaps surprising since the genetic class is defined only from the breeding value of the father; a combination of the breeding values of the father and the mother should show an even larger effect. Even in the present model the culling of cows of low genetic class is much more intensive than for cows of high genetic class.

It is also possible to include information about the heifers available with respect to genetic class. Table III is based on a comparison of the present value if the cow is kept with the expected present value of an unknown replacement

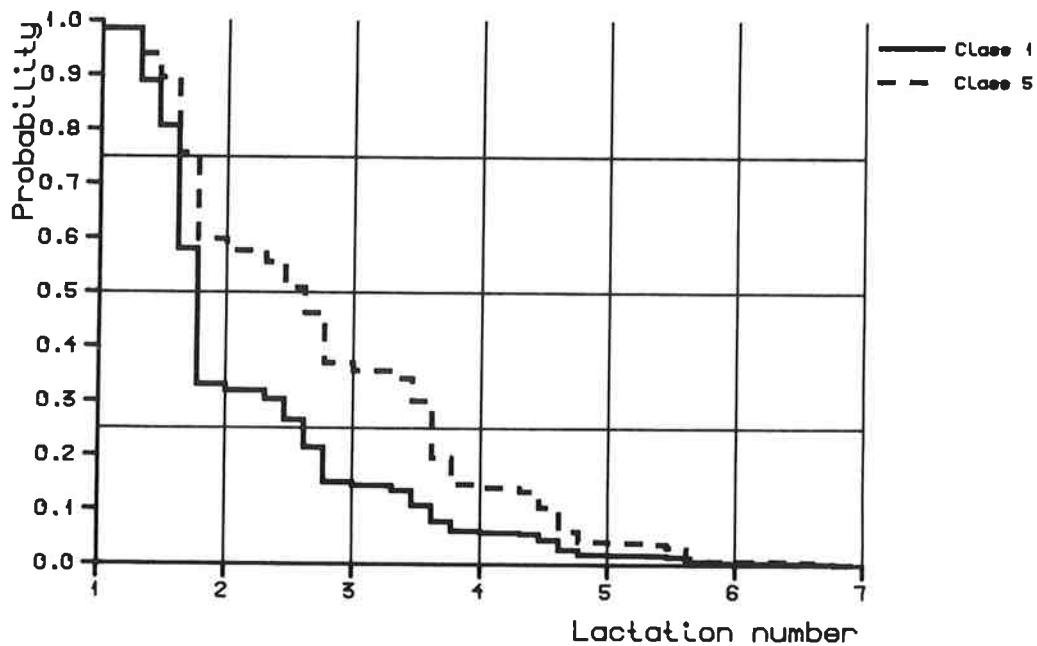


Fig. 1. Survival curve of the lowest genetic class (Class 1) compared to the corresponding curve of the highest genetic class (Class 5). The curves give for each class the probability that a heifer having entered the herd at first calving is still present in the herd at the time indicated.

heifer (i.e. unknown genetic class). If the expected present value of an unknown heifer is denoted  $r$  and the present value of a heifer from the  $i$ th genetic class is denoted  $r_i$  we can define the relative present value of such heifer as  $d_i = r_i - r$ . Table IV shows the relative present values of heifers from the various genetic classes, and together with Table III, gives useful additional information. If the future profitability of state  $j$  (in Table III  $j$  can take  $2 \times 15^2$  values) is denoted  $p_j$ , and if  $d_{i-1} < p_j \leq d_i$  then a cow in state  $j$  should be replaced if a heifer from at least the  $i$ th genetic class is available. If the best heifer available is from the genetic class  $i-1$  or lower, it is better to sell the heifer and keep the cow. In this way the heifers are included in the ranking because the least profitable cows should be replaced, even if only by heifers from the lowest genetic class. A group of cows follows which can be replaced by heifers from the second genetic class (or higher) and so on until the highest ranking cows which should not be replaced even by the best heifers.

TABLE IV

Relative present values of heifers from the various genetic classes (Dkr.)

Genetic class				
1	2	3	4	5
-395	-290	-40	282	526

## CONCLUSIONS

It has been shown that it is possible to model the dairy cow replacement problem within the theory of hierarchic Markov processes. By this method exact solutions are given to a model which would contain more than 60 000 states if it were to be formulated as an ordinary Markov decision process. Optimal actions are determined and the cows of a herd are ranked according to their future profitability derived from the optimal solution.

The most important state variable is the milk yield of the present lactation, but there is also a considerable influence of the calving interval on the optimal actions. On the other hand the influence of the milk yield of the previous lactation was rather small, in particular when the genetic class was taken into account. It is concluded that in a model which contains the milk yield of the present lactation and the genetic class as state variables it is hardly necessary to include the milk yield of the previous lactation as a state variable.

The genetic class, defined as the breeding value (concerning milk yield) of the father, was shown to exert a considerable influence on average herd life through the actions taken. This variable made it possible to include heifers in the ranking of cows because the breeding value of the father (i.e. if the father is a proven sire) is usually known prior to the first calving. It must be assumed that the genetic class would be even more important if it was defined from the breeding value of both the mother and the father. This was not possible in the present study because of lack of data. If this effect is wanted in a model genetic theory could be used to predict the influence on the daughter's production.

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## RESUME

Kristensen, A.R., 1987. Renouvellement optimal et classement des vaches laitières a partir d'un processus hiérarchique de Markov. *Livst. Prod. Sci.*, 16: 131-144 (en anglais).

On a élaboré un modèle pour renouvellement des vaches laitières sur la base d'un processus hiérarchique de Markov. Ce dernier est une série de processus de décision de Markov assemblés dans un processus de Markov appelé le processus principal.

Dans le modèle, les vaches sont décrites par leur numéro et leur stade de lactation, par le niveau de leur production laitière dans la lactation antérieure et la lactation en cours, par la durée de l'intervalle entre vêlages et par leur classe de valeur génétique calculée à partir de celle de leur père. L'objectif est de maximiser la valeur actuelle du revenu net d'une période infinie. On a pris en considération les revenus apportés par le lait, le veau et la vache lors de sa réforme et les coûts des aliments et des génisses de remplacement.

Le profit à venir, qui est déterminé par la solution optimale, est appliqué au classement des vaches dans le troupeau. La classe de valeur génétique offre la possibilité d'inclure les génisses de remplacement disponibles dans le classement et de laisser les décisions de remplacement dépendre de la classe de valeur génétique des génisses.

On conclut que la production laitière de la lactation antérieure n'est pas nécessaire quand les autres variables sont dans le modèle.

## KURZFASSUNG

Kristensen, A.R., 1987. Optimale Remontierung und Rangierung von Milchkühen durch einen hierarchischen Markov-Prozess. *Livst. Prod. Sci.*, 16: 131-144 (auf englisch).

Es wird ein Modell zur Remontierung von Milchkühen vorgestellt, das als hierarchischer Markov-Prozess formuliert ist. Ein hierarchischer Markov-Prozess ist als Serie von Markov-Entscheidungsprozessen zu verstehen, die in einem Markov-Entscheidungsprozess (Hauptprozess) zusammengefasst sind.

In dem Modell wird eine Kuh gekennzeichnet durch Laktationsnummer, Laktationsstadium, Niveau der Milchleistung in der vorangegangenen und laufenden Laktation, Zwischenkalbezeit und genetischer Klasse (definiert durch den Vater-Zuchtwert). Als Optimierungskriterium wird der diskontierte Nettogewinn bei unendlichem Planungshorizont maximiert. Dabei werden Erträge aus Milch, Kälbern und Schlachtkühen sowie Kosten für Futter und Bestandsergänzung berücksichtigt.

Zur Rangierung der Kühe innerhalb Herde wird ihr erwarteter Betriebswert aus der optimalen Lösung verwendet. Durch die Verwendung der genetischen Klassen können auch die zur Remontierung bereitstehenden Jungrinder in die Rangierung einbezogen werden.

Die Ergebnisse zeigen, dass die Milchleistung der vorangegangenen Laktation nicht herangezogen werden muss, solange die übrigen Informationen einbezogen sind.

V



# Maximization of net revenue per unit of physical output in Markov decision processes\*

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## Summary

*A new criterion of optimality in Markov decision processes is discussed. The objective is to maximize the average net revenue per unit of physical output (or input). The criterion is relevant in some production models where a limitation is imposed on the physical output (production quota) or on an input factor (scarce resources). An obvious application is in dairy cow replacement models under milk quotas. Iteration cycles are presented for ordinary completely ergodic Markov decision processes and for hierarchic Markov processes. The consequences of the new criterion are illustrated by a numerical example.*

*Keywords: Markov decision process, optimization, milk quota, replacement.*

## 1. Introduction

Consider a system which at any time is described by its *state* defined by all relevant information on the characteristics of the system. The state is observed at discrete time intervals called *stages*, and the transition from one state to another at the following stage is governed by probability distributions depending on the present state, but, independent of the states previously observed. It is assumed that a *reward* (depending on the state) is gained in each stage. The system may be affected by *actions* which influence the reward of the present stage and the probability distribution of the state to be observed at the next stage.

A *stochastic process* like the one described above is called a *Markov decision process*, which is an important tool in operations research. The original formulation is due to Howard (1960). Several techniques are avail-

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able for optimization, i.e., for each possible state to decide which action to choose in order to maximize a predefined objective function giving the criterion of optimality. The literature on Markov decision processes is extensive, but careful reviews have been given by van der Wal and Wessels (1985) as well as White and White (1989). The numerous applications in agriculture have been discussed and reviewed by Kennedy (1986).

One of the most successful areas of application in agriculture has been in animal replacement problems. The earlier examples in dairy cows have been reviewed by van Arendonk (1984). More recent applications are van Arendonk (1986) and Kristensen (1987, 1988). In these models each cow and its future successors represent a separate Markov decision process. The states are defined from the characteristics of the present cow (i.e., lactation number, stage of lactation, reproductive status, milk yield etc.). The actions of the models are to keep the cow or to replace it. Similar studies on sow replacement have been carried out by Huirne et al. (1988) and Dijkhuizen et al. (1989).

The usual criterion of optimality for Markov decision processes running over an infinite number of stages has been either the average rewards criterion, where the average of rewards per stage is maximized, or the discounting criterion, where the total of expected discounted rewards (i.e., the expected present value) is maximized. In some applications, however, neither of these will suffice. A typical case where the need for an alternative criterion emerges, is when production is modelled in a situation where some restriction is imposed on the system. The restriction could either be a limitation in output (production quota) or a limited supply of some input factor used in the production (scarce resources).

As an example, consider the dairy cow replacement models mentioned above (Kristensen 1987, 1988 or van Arendonk 1986). Since a dairy farmer can only have some fixed maximum number of cows, an appropriate criterion of optimality (if no further restrictions are imposed) is the maximization of net revenue per cow in the long run. Since each cow represents a separate Markov decision process, that implies the usual discounting criterion. Suppose now, that a production quota is imposed that limits the amount of milk that the individual farmer is allowed to produce. (Such quotas have been introduced in all countries of the EC.) If the quota is obtainable with fewer cows than the maximum set by housing capacity, a better criterion of optimality is obviously the maximization of average net revenue per kg of milk produced.

In the present paper such a criterion of optimality is discussed, and methods to provide optimal policies are presented for ordinary Markov decision processes as well as the special hierarchic Markov processes described by Kristensen (1988). For an application of the new criterion of optimality in a dairy cow replacement model, reference is made to Kristensen (1989).

## 2. Ordinary completely ergodic Markov decision processes

Consider a discrete time Markov decision process with a finite *state* space  $\Omega = \{1, \dots, w\}$  and a finite *action* set  $D$ . A policy  $s$  is a map assigning to each state  $i$  in  $\Omega$  an action  $s(i)$  from  $D$ . Let  $p_{ij}^d$  be the *transition probability* from state  $i$  to state  $j$  if the action  $d$  is taken. As usual, all transition probabilities are non-negative and  $\sum_{j=1}^w p_{ij}^d = 1$  for all  $i, d$ . The reward to be gained when the state  $i$  is observed, and action  $d$  is taken, is denoted  $r_i^d$ . The time interval between two transitions is called a *stage*. If  $s(i) = d$ , the symbols  $r_i^d$  and  $p_{ij}^d$  are also written as  $r_i^s$  and  $p_{ij}^s$  respectively.

If the traditional average rewards criterion is used we search for a policy that maximizes

$$g_r^s = \sum_{i=1}^w \pi_i^s r_i^s, \quad (1)$$

where  $g_r^s$  is the average rewards per stage and  $\pi^s$  is the limiting state distribution of the Markov process under the policy  $s$  (i.e., when the policy is kept constant over an infinite number of stages). We assume that the Markov process is completely ergodic for any policy, i.e., that a unique limiting state distribution exists.

We have now defined the elements of a traditional Markov decision process, but in this paper we further assume that if state  $i$  is observed, and action  $d$  is taken, a physical quantity of  $m_i^d$  is involved. The quantity  $m_i^d$  can either be an output factor or an input factor. In the case of a dairy cow replacement model,  $m_i^d$  would be the amount of milk produced by a cow in state  $i$  when action  $d$  is taken. In this paper we shall refer to  $m_i^d$  as the *physical output*. It is obvious that we can calculate the average physical output  $g_m^s$  under the policy  $s$  in a way analogous with equation (1):

$$g_m^s = \sum_{i=1}^w \pi_i^s m_i^s. \quad (2)$$

In this paper an optimal policy is defined as a policy that maximizes

$$g^s = g_r^s / g_m^s = \left( \sum_{i=1}^w \pi_i^s r_i^s \right) / \left( \sum_{i=1}^w \pi_i^s m_i^s \right), \quad (3)$$

where  $g^s$  is the average rewards per unit of physical output when the policy is kept constant over an infinite number of stages. To ensure that  $g^s$  is always defined (and that the sign is determined by  $g_r^s$ ) we assume, that for all  $s$ ,

$$g_m^s = \sum_{i=1}^w \pi_i^s m_i^s > 0.$$

Fortunately we can profit from results obtained in the area of semi-Markov decision programming, where the average net revenue over time has been

handled by Jewell (1963) (and others) for processes where stage length is a stochastic variable whose distribution depends on the state  $i$  and the action  $d$ . Denote as  $t_i^s$  the expected stage length given  $s$  and  $i$ . Jewell (1963) presented an algorithm to find a policy that maximizes

$$g^s = \left( \sum_{i=1}^w \pi_i^s r_i^s \right) / \left( \sum_{i=1}^w \pi_i^s t_i^s \right). \quad (4)$$

By comparing equations (3) and (4) it is seen that the two problems are completely analogous. Thus we can apply the algorithm of Jewell to the present case, where we want to find a policy that maximizes  $g^s$  of equation (3). The iteration cycle to be used can be stated as follows:

- 1) Choose an arbitrary policy  $s$ . Go to 2.
- 2) Solve the set of  $w+1$  simultaneous equations for  $g^s$  and the *relative values* (cf. Section 3)  $f_1^s, \dots, f_w^s$ :

$$g^s m_i^s + f_i^s = r_i^s + \sum_{j=1}^w p_{ij}^s f_j^s, \quad i = 1, \dots, w \quad (5)$$

$$f_w^s = 0. \quad (6)$$

Go to 3.

- 3) For each state  $i$ , find the alternative  $d'$  from  $D$  that maximizes

$$r_i^d - m_i^d g^s + \sum_{j=1}^w p_{ij}^d f_j^s$$

and let  $s'(i) = d'$ . If  $s' = s$ , then stop, since an optimal policy is found. Otherwise, redefine  $s$  according to the new policy (i.e., let  $s = s'$ ) and go back to 2.

It should be noted that by equations (5) and (6) we have a general set of equations to determine  $g^s$  under any predefined policy. Thus we are able to calculate the economic consequences of using alternative non-optimal policies instead of an optimal policy. In this way the benefits of the very model may be evaluated. Kristensen and Thysen (1991) have used this approach in order to evaluate the value of culling information in commercial dairy herds in the presence and absence of a milk quota.

The equations are also useful in calculating other technical results under a given policy by redefining  $r_i^s$  and  $m_i^s$ . A few examples from a dairy cow replacement model are:

- 1) If  $r_i^s$  is the milk yield of a cow in state  $i$  under policy  $s$ , and  $m_i^s$  is the stage length when state  $i$  is observed under policy  $s$ , then  $g^s$  is the average milk yield per cow per year under policy  $s$ .

- 2) Let  $r_i^s = 1$  if state  $i$  represents the purchase of a heifer and zero otherwise. Let  $m_i^s$  be defined as in 1. Then  $g^s$  is the annual replacement rate under  $s$ .
- 3) Let  $r_i^s = 1$  if a calving takes place and zero otherwise. If  $m_i^s$  is defined as in 1, then  $g^s$  is the average number of calvings per cow per year under  $s$ .
- 4) Let  $r_i^s = n$  and  $m_i^s = 1$  if a calving takes place after a calving interval of  $n$  weeks, and both zero otherwise. Then  $g^s$  is the average length of the calving interval under  $s$ .

### 3. Interpretation of relative values

The unknowns  $f_1^s, \dots, f_w^s$  in the equations in Step 2 of the iteration cycle of Section 2 are called the *relative values* of the states in  $\Omega = \{1, \dots, w\}$ . In this section we shall discuss the practical interpretation of the relative values. For convenience we shall restrict ourselves to the ordinary completely ergodic Markov processes of Section 2, but similar interpretations are easily given in the hierarchic case of Section 4.

Define  $R_i^s(w)$  as the total expected reward until state  $w$  is reached the next time under the policy  $s$  when the present state is  $i$ . Similarly  $M_i^s(w)$  is the analogous total expected physical output. By simple but tedious arguments it can be shown that

$$f_i^s = R_i^s(w) - g^s M_i^s(w). \quad (7)$$

The practical interpretation of equation (7) is that in equation (6) state  $w$  is defined as the zero state. If we start from any other state  $i$ , the process will differ from a situation where we start from state  $j \neq i$  until state  $w$  is reached the first time. In state  $w$  the relative value  $f_w^s$  is zero, and in front of us we have an infinite process where the average rewards per unit of physical output are  $g^s$ . Thus the whole difference in profitability between state  $i$  and state  $j$  concerns the initial period until state  $w$  is reached the first time. If  $R_i^s(w) > g^s M_i^s(w)$ , it means that the average rewards per unit of physical output in the initial period are larger than  $g^s$ . Thus the relative value of state  $i$  is larger than  $f_w^s = 0$ . Similarly, if  $R_i^s(w) < g^s M_i^s(w)$ , we have that  $f_i^s < f_w^s = 0$ . In words, the relative value  $f_i^s$  equals the amount of money that a rational person is just willing to pay in order to start in state  $i$  rather than state  $w$ .

The relative values are useful when we want to investigate the consequences of deviating from the optimal policy. Suppose that we have determined the optimal policy  $s$ . In a replacement model, for instance, the typical actions are 'keep' and 'replace'. For convenience we assume that only these two actions (referred to as '1' and '2' respectively) are possible. At the present stage we consider for some reason to deviate from the policy  $s$ , but

during the following stages we intend to follow  $s$ . If we take action 1 at the present stage, and the state is  $i$ , the relative value is

$$f_i^1 = r_i^1 - m_i^1 g^s + \sum_{j=1}^w p_{ij}^1 f_j^s,$$

and if we take the action 2, the corresponding relative value is

$$f_i^2 = r_i^2 - m_i^2 g^s + \sum_{j=1}^w p_{ij}^2 f_j^s.$$

Thus the difference

$$\delta_i^s = f_i^1 - f_i^2 \tag{8}$$

is equal to the gain from taking the action 1 rather than 2. If  $\delta_i^s$  is negative, it means that it is better to take action 2 than 1. In a replacement model the difference  $\delta_i^s$  can be used for ranking of states, starting with negative  $\delta_i^s$ , where replacement is optimal, and ending with large positive  $\delta_i^s$ , where it is certainly most profitable to keep. In such a model  $\delta_i^s$  is called the *future profitability* of state  $i$  under the policy  $s$ . For an example of the application of future profitabilities in a dairy cow replacement model under milk quotas reference is made to Kristensen (1989).

#### 4. Hierarchic Markov processes

##### 4.1. Notation and problem formulation

In Kristensen (1988) a new notion of a hierarchic Markov process was introduced. We shall now describe how the new criterion of optimality is used with such processes.

A hierarchic Markov process is a series of Markov decision processes called *subprocesses* built together in one Markov decision process called the *main process*. A subprocess is a discrete time Markov decision process with  $N$  stages and a finite state space  $\Omega_n = \{1, \dots, w_n\}$  for stage  $n$ ,  $1 \leq n \leq N$ . The action set  $D_n$  of the  $n$ 'th stage is assumed to be finite, too. A policy  $s$  of a subprocess is a map assigning to each stage  $n$  and state  $i$  in  $\Omega_n$  an action  $s(n, i)$  from  $D_n$ . The set of all possible policies of a subprocess is denoted  $\Gamma$ . When the state  $i$  is observed at stage  $n$  and the action  $d$  is taken, a reward  $r_i^d(n)$  is gained. The corresponding physical output is denoted  $m_i^d(n)$ . Let  $p_{ij}^d(n)$  be the transition probability from state  $i$  to state  $j$  where  $i$  is the state of the  $n$ 'th stage,  $j$  is the state of the following stage, and  $d$  is the action taken at stage  $n$ . In this formulation the transition matrix  $P_n^s = \{p_{ij}^s(n)\}$  is not necessarily a square matrix. Naturally  $p_{ij}^s(n) \geq 0$  for all  $i, j, n$  and  $s$ , and for all  $i, n$  and  $s$

$$\sum_{j=1}^{w_{n+1}} p_{ij}^s(n) = 1.$$

The vector  $P_0 = (p_1(0), \dots, p_{w_1}(0))'$  gives the probability distribution of the states at stage 1. As before, the parameters may be superscripted by actions or policies as convenient.

The total expected reward function  $f_i^s(n)$  represents the total expected rewards from the rest of the process when the present state and stage are  $i$  and  $n$  respectively, and the policy  $s$  is followed. The function is defined as follows for given  $s$ ,  $i$  and  $n$ :

$$\begin{aligned} f_i^s(n) &= r_i^s(n), \quad n = N \\ f_i^s(n) &= r_i^s(n) + \sum_{j=1}^{w_{n+1}} p_{ij}^s(n) f_j^s(n+1), \quad n = N-1, \dots, 1. \end{aligned} \quad (9)$$

The total expected output function  $h_i^s(n)$  is defined completely analogously with equation (9). The only difference is that  $f$  is replaced by  $h$ , and  $r$  by  $m$ .

Assume that we have a set of  $v$  possible subprocesses. The main process is then a Markov decision process with an infinite number of stages and the finite state space  $\{1, \dots, v\}$ . Each stage in this process represents a particular subprocess. The action sets of the main process are the sets  $\Gamma_\alpha$ ,  $\alpha = 1, \dots, v$ , of all possible policies of the individual subprocesses (to avoid ambiguity the states of the main process will be denoted by Greek letters  $\alpha$ ,  $\beta$  etc.). A policy  $\sigma$  is a map assigning to each state  $\alpha$  of the main process an action  $\sigma(\alpha)$  from  $\Gamma_\alpha$ . The transition matrix of the main process has the dimension  $v \times v$ , and it is denoted  $\Phi = \{\phi_{\alpha\beta}\}$ . The transition probabilities are usually assumed to be independent of the action taken. The reward  $f_\alpha^\sigma$  in state  $\alpha$  of the main process is defined from the function  $f_i^s$  of the  $\alpha$ 'th subprocess:

$$f_\alpha^\sigma = \sum_{i=1}^{w_1} p_i(0) f_i^s(1), \quad s = \sigma(\alpha), \quad (10)$$

where the parameters of the right hand side of equation (10) are those belonging to the  $\alpha$ 'th subprocess. Similarly the physical output in state  $\alpha$  is given as

$$h_\alpha^\sigma = \sum_{i=1}^{w_1} p_i(0) h_i^s(1), \quad s = \sigma(\alpha). \quad (11)$$

Since the main process is an ordinary Markov decision process, the iteration cycle described in Section 2 can in principle be used directly for optimization. In practice Steps 1 and 2 are easily carried out, but Step 3 is prohibitive because of the extremely large number of alternative actions  $s$  in  $\Gamma_\alpha$  (as mentioned above  $s$  is an entire policy of the  $\alpha$ 'th subprocess). Thus an alternative version of Step 3 is needed. In the following section we shall discuss a method to circumvent this problem.

## 4.2 Optimization

Using the notation of the main process, Step 3 of the iteration cycle involves finding the alternative  $s'$  that maximizes

$$f_\alpha^s - h_\alpha^s g^\sigma + \sum_{\beta=1}^v \phi_{\alpha\beta} F_\beta^\sigma, \quad (12)$$

where  $F_\beta^\sigma$  is the relative value of state  $\beta$  in the main process under the policy  $\sigma$ . Since the last term of the expression does not depend on  $s$ , the problem readily reduces to finding the alternative, that maximizes

$$f_\alpha^s - h_\alpha^s g^\sigma. \quad (13)$$

The reduction from equation (12) to (13) is computationally convenient, and in fact this causes a more simple optimization cycle than under the discounting criterion as described by Kristensen (1988). The symbols  $f_\alpha^s$  and  $h_\alpha^s$  refer to the total expected rewards and total expected physical outputs, respectively, during the entire subprocess  $\alpha$  under the policy  $s$ .

Now, define a new Markov decision process with the same state and action spaces and the same transition probabilities as the  $\alpha$ 'th subprocess. The rewards of the new process are defined as

$$a_i^d(n) = r_i^d(n) - m_i^d(n) g^\sigma. \quad (14)$$

The new process is defined to run over  $N$  stages just as the original subprocess. It is easily seen that a policy that maximizes the total expected rewards from the new process also will maximize the expression (13). Thus such a policy can be found by means of the well known value iteration method using the recurrence equations

$$\begin{aligned} A_i(n) &= \max_d \{a_i^d(n)\}, \quad n = N \\ A_i(n) &= \max_d \left\{ a_i^d(n) + \sum_{j=1}^{w_{n+1}} p_{ij}^d(n) A_j(n+1) \right\}, \quad n = N-1, \dots, 1. \end{aligned} \quad (15)$$

The actions defining the policy  $s$  are defined step by step choosing in each state the one that maximizes the right hand side of equation (15). The symbol  $A_i(n)$  denotes the total expected rewards from the remaining part of the new process, when it presently is at the beginning of stage  $n$ .

If we use the recurrence equations (15) in Step 3 of the iteration cycle, we arrive at the following formulation if we change the notation back to that of the subprocess:

- 1) Choose an arbitrary policy  $\sigma$ . Go to 2.
- 2) Solve the following set of  $v+1$  linear simultaneous equations for  $g^\sigma$  and  $F_1^\sigma, \dots, F_v^\sigma$ :

$$g^\sigma h_\alpha^\sigma + F_\alpha^\sigma = f_\alpha^\sigma + \sum_{\beta=1}^v \phi_{\alpha\beta} F_\beta^\sigma, \quad \alpha = 1, \dots, v \quad (16)$$

$$F_v^\sigma = 0 \quad (17)$$

Go to 3.

3) For each subprocess  $\alpha$ , find by means of the recurrence equations

$$\tau_{\alpha,i}(n) = \max_d \{r_i^d(n) - m_i^d(n)g^\sigma\}, n = N$$

$$\tau_{\alpha,i}(n) = \max_d \left\{ r_i^d(n) - m_i^d(n)g^\sigma + \sum_{j=1}^{w_{n+1}} p_{ij}^d(n) \tau_{\alpha,j}(n+1) \right\}, n = N-1, \dots, 1.$$

a policy  $s'$  of the subprocess. The action  $s'(n, i)$  is equal to the  $d'$  that maximizes the right hand side of the recurrence equation of state  $i$  at stage  $n$ . Put  $\sigma'(\alpha) = s'$  for  $\alpha = 1, \dots, v$ . If  $\sigma' = \sigma$ , then stop, since an optimal policy is found. Otherwise redefine  $\sigma$  according to the new policy (i.e., put  $\sigma = \sigma'$ ) and go back to 2.

In large models it may be difficult to choose a relevant initial policy in Step 1. In that case the cycle may be initiated in Step 3 by choosing an initial value for  $g^\sigma$ .

Just as equations (5) and (6) could be used in a general way to calculate the economic consequences and a variety of technical results under any policy in the case of an ordinary Markov process, exactly the same applies to the equations of Step 2 of the hierarchic algorithm.

## 5. The new criterion compared to others

In the literature two different criteria of optimality (the discounting and the average rewards criterion) have been considered for infinite stage Markov decision processes. Optimization cycles for both were described already by Howard (1960) for ordinary Markov decision processes.

The criterion most frequently used in animal replacement models is the discounting criterion where all rewards are discounted to the beginning of the planning horizon and the total *present value* is maximized. If the discount factor is less than unity, the total present value over an infinite number of stages will be finite, and an optimal policy exists. No physical output  $m_i^s$  is considered. An optimization cycle for the hierarchic case was developed by Kristensen (1988).

Under the average rewards criterion the optimization cycle ensures that the *average rewards per stage* are maximized. Provided that all stages are of equal duration, this is the same as maximization of average rewards over time. No physical output is considered and no optimization cycle has been published in the hierarchic case. Furthermore no discounting takes place.

The criterion used in this paper is in fact a generalization of the average rewards criterion. If  $m_i^s = 1$  for all  $s$  and  $i$  (or  $m_i^s(n) = 1$  for all  $s, i$  and  $n$  in the hierarchic case) the criterion of this paper reduces to the average rewards criterion. Thus the hierarchic cycle of Section 4 may be used also under the average rewards criterion just by setting all physical outputs to unity.

The main difference between the criterion of this paper compared to the discounting criterion as presented by Kristensen (1988) is that in the latter a *sum* of expected discounted rewards is maximized, whereas in this paper a long run average reward/output *ratio* is maximized. Furthermore the equations (5)–(6) and (16)–(17), which are used in the optimization cycles under the present criterion, are useful in the calculation of a variety of other technical ratios that may be used to investigate the consequences of any policy as discussed in Section 2. Because no discounting is involved, the optimization cycle is also more simple in the hierarchic case under the present criterion compared to the discounting criterion discussed by Kristensen (1988).

In order to illustrate the differences between the three criteria we shall consider a simple numerical example, and for convenience we use an ordinary Markov decision process.

Consider a multicomponent production system (e.g., a dairy herd consisting of individual cows). We shall determine optimal replacement policies of the individual components under the three criteria.

Assume that each component gives rise to two different kinds of output items (1 and 2, e.g., milk and calves). We shall represent each component by a Markov decision process, where the states are defined from the level of performance in producing output item 1, and we assume that the production of item 2 is independent of state. For convenience we shall only consider three states called 'bad', 'normal' and 'good' each representing the production of 3, 4 and 5 units of output item 1, respectively. Each component is inspected at regular time intervals (stages) in order to determine the state of operation. After inspection we can choose to keep the component for at least one additional stage, or we can replace it at the end of the stage at some additional cost.

If we replace, the new component enters one of the three states with equal probability. If the component is kept, the probability of staying in the present state is 0.6, and if the present state is 'normal' the probability of transition to one of the other states is 0.2 each. The probability of transition (if kept) from 'bad' or 'good' to 'normal' is, in both cases 0.3, and from 'bad' to 'good' and vice versa the probability is 0.1.

The relation between the rewards (net revenues) and physical outputs (item 1) is assumed to be as follows (decision 1 = 'keep' and 2 = 'replace'):

$$r_i^1 = c_1 m_i^1 + c_2 n - c_3, \quad i = 1, 2, 3 \quad (18)$$

and

$$r_i^2 = c_1 m_i^2 + c_2 n - c_3 - c_4, \quad i = 1, 2, 3, \quad (19)$$

where  $c_1$  and  $c_2$  are the marginal net returns of output item 1 and 2, respectively,  $n$  is the production of item 2,  $c_3$  is the fixed cost per stage of operating a component and  $c_4$  is the replacement cost. Since replacement is

carried out at the end of a stage, we assume that  $m_i^1 = m_i^2$  for all  $i$ . The resulting parameters of the Markov decision process assuming  $c_1 = 1$ ,  $c_2n - c_3 = 2$  and  $c_4 = 0.5$  are summarized in Table 1.

Optimal replacement policies were determined under all three criteria and the results are shown in Table 2, which gives the optimal actions, the associated relative values of states and the values of all three objective functions under the optimal policies. In the following we shall denote as, for example,  $r-k-k$  a policy defined as replacing in state 'bad' and keeping in states 'normal' and 'good', respectively.

As it appears the discounting and the average rewards criteria give rise to the same optimal policy  $r-k-k$  (and thus the same values of the objective functions). Furthermore the relative values of states are almost equal. The similarity is probably caused by the infinite horizon combined with the recurrent nature of the process (i.e., the state presently observed will sooner or later be observed again). Under such conditions the two criteria express almost the same. It should be noticed from the relative values under these criteria, that the names 'bad', 'normal' and 'good' of the states are quite suitable.

Table 1. *Parameters of the Markov decision process of the numerical example*

State <sup>1</sup>	Keep					Replace				
				$m_i^1$	$r_i^1$				$m_i^2$	$r_i^2$
	$j=1$	$p_{ij}^1$ $j=2$	$j=3$			$j=1$	$p_{ij}^2$ $j=2$	$j=3$		
$i=1$	0.6	0.3	0.1	3	5	1/3	1/3	1/3	3	4.5
$i=2$	0.2	0.6	0.2	4	6	1/3	1/3	1/3	4	5.5
$i=3$	0.1	0.3	0.6	5	7	1/3	1/3	1/3	5	6.5

1. States 1, 2, and 3 are 'bad', 'normal' and 'good', respectively.

Table 2. *Optimal replacement policies (k=keep, r=replace, kr=keep or replace), optimal relative values of states and values of the objective functions using the parameters of Table 1 under three different criteria of optimality*

Criterion	Relative values <sup>1</sup>			Present value	Av. rew. per	
	Bad	Normal	Good		Stage	Output
Discounting	$\gamma$ 3.24 <sup>r</sup>	-1.77 <sup>k</sup>	0.00 <sup>k</sup>	<u>60.65</u>	6.094	1.455
Av. reward/stage	-3.34 <sup>r</sup>	-1.91 <sup>k</sup>	0.00 <sup>k</sup>	60.65	<u>6.094</u>	1.455
Av. reward/output	2.00 <sup>k</sup>	1.00 <sup>k</sup>	0.00 <sup>kr</sup>	57.30 <sup>2</sup>	5.719 <sup>2</sup>	<u>1.500</u>

1. Under the discounting criterion the relative value is defined as the present value of state  $i$  minus the present value of state 3 ('good').

2. If 'replace' in state 'good', otherwise 60.00 and 6.000.

The criterion of the present paper (average rewards/output ratio) is fundamentally different in its consequences. In this situation two policies are optimal:  $k-k-k$  and  $k-k-r$ . Further the relative values of states 1 and 2 are now higher than that of state 3, so in fact the names 'good' and 'bad' should really be interchanged under this criterion. The criterion is relevant if a production quota is imposed on output item 1. In that case the total net revenue from production would equal  $1.500 Q$ , where  $Q$  is the size of the quota, under the present criterion. If the optimal policy of one of the other criteria was used, the total net revenue from production would only amount to  $1.455 Q$ . In both cases the number of components in operation should be adjusted to meet the quota. The number of items produced by a component per stage under a policy is calculated as (average rewards/stage)/(average rewards/output), i.e.,  $5.719/1.500 = 3.813$  under policy  $k-k-r$ , and  $6.094/1.455 = 4.188$  under policy  $r-k-k$ .

In this very simple example, there were really no need for optimization cycles since the total number of policies is only 8. We could just have calculated the value of all three objective functions under all policies, and for each criterion choose the policy that maximizes the relevant objective function. As an additional illustration of the different consequences of policies under the three criteria these calculations were carried out. The policy ranking lowest was in all cases  $r-r-r$ . The values of the objective functions were defined as 100 for this policy, and other policies were expressed relatively to that.

In Figure 1 the relative ranking of all policies is shown under the three

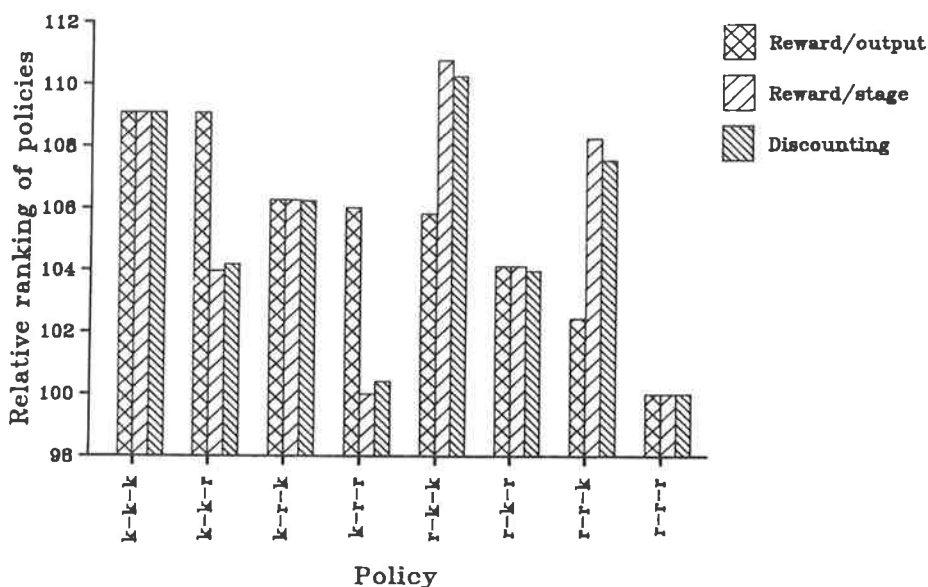


Figure 1. *Relative ranking of policies under the three criteria of optimality (policy  $r-r-r = 100$ )*

criteria. Again the discounted and the average rewards criteria show almost exactly the same pattern over policies, while the pattern of the average rewards/output criterion is very different.

## 6. Discussion and applications

It is an accepted fact that profitability must be maximized from the most limiting input or output factor. Therefore the original purpose of the work behind this paper was to become able to maximize net revenue per kg milk produced in a dairy cow replacement model to be used under milk quotas (Kristensen 1989).

The model is of the hierarchic type, and the iteration cycle of Section 4 has successfully been used for optimization. The total number of state combinations  $(\alpha, i)$  of the model equals 180080. In a test, an optimal solution was calculated in 100 cases representing different price and production conditions. The number of iterations required ranged from 3 to 6. If the number of iterations is compared to the size of the model we can conclude that the iteration cycle has proved to be very efficient in practical applications.

Regardless of the original purpose of the iteration cycles of this paper they are believed to be useful also in other areas than dairy cow replacement under milk quotas. The problem of production under limitations is general, and anywhere a Markov decision process is used for optimization the iteration cycles of this paper may be relevant.

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## Optimal Replacement and Ranking of Dairy Cows under Milk Quotas

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The usual criterion of optimality to be used in dairy cow replacement models has typically been the maximization of total discounted net revenue per cow. In a situation with herd individual milk quotas, however, the theoretically correct criterion is the maximization of net revenue per kg of milk produced. Optimal replacement policies, future profitabilities, and rankings under the two criteria are compared. It turns out that culling should be less intensive under milk quotas because of a smaller variation in future profitability. Considerable differences in future profitability and ranking are found, and it is concluded that it is important that the correct criterion is used when milk quotas are in effect.

### INTRODUCTION

Recently the EEC has introduced herd individual milk quotas in all member countries which impose restrictions on the amount of milk to be produced per year. Before the quotas were introduced, the limiting restriction was typically a maximum herd size set by stable capacity. The new situation may affect the optimal dairy cow replacement policy. A common trait of recent replacement models (van Arendonk, 1985, 1986; van Arendonk & Dijkhuizen, 1985; Kristensen, 1986, 1987a, 1988) is that the criterion of optimality is the maximization of the expected present value of future rewards (discounting criterion). Since the models handle one cow and its future successors this is equal to the maximization of net revenue per cow.

Under milk quotas this criterion is no longer appropriate since economic efficiency in general should be expressed in terms of the most limiting restriction (Barnard & Nix, 1973). A better criterion is the maximization of net revenue per kg of milk produced (average criterion). The aim of the present paper is to introduce this criterion in a dairy cow replacement model based on dynamic programming. Furthermore the paper will investigate to what extent the criterion of optimality influences the optimal replacement and ranking of dairy cows, or in other words, whether the replacement and ranking of dairy cows depend on the presence or absence of milk quotas. It is not the purpose of the paper to discuss the application of replacement models in general or to evaluate dynamic programming as a tool in this context. For such discussion reference is made to van Arendonk (1984) and to the models mentioned above. Neither is it the purpose to discuss a general adjustment to milk quotas (by e.g. reduced feeding). It is assumed that feeding and biological parameters are the same in both situations and that only the replacement policy is changed.

### MATERIAL AND METHODS

#### *The model*

The model to be used in this study is based on a hierarchic Markov process as it is described by Kristensen (1988). The idea of such a model is that a series of Markov

decision processes called subprocesses are built together in one Markov decision process called the main process. The structure is specially designed to fit replacement models where considerable computational advantages are obtained in large models. For a discussion of Markov decision processes in general, reference is made to Howard (1960).

The basic biological parameters to be used in the model are published by Kristensen (1986). In the model decisions concerning replacement are made regularly every fourth week from first calving of a heifer until 40 weeks after sixth calving. Changes in production level and state of pregnancy are assumed always to take place according to probability distributions at 16, 24, 32, or 40 weeks after a calving. In the periods between these points the milk yield is predicted in a deterministic manner.

State variables are defined in the main process as well as in the subprocesses. In the main process the only state variable is the genetic class (5 levels) defined from the relative breeding value of the father as described by Kristensen (1986). In the subprocesses (there is one subprocess for each genetic class) the state variables are the milk yield of previous and present lactation (15 classes each) and the length of the calving interval (8 classes: 44, 48, ..., 72 weeks where "44 weeks" covers all intervals up to 46 weeks, "48 weeks" covers intervals more than 46 weeks up to 50 weeks etc.). The classes of milk yield are defined from the milk yields in kg 4% milk adjusted for lactation number, stage of lactation, herd level, length of calving interval, and the genetic class, as described by Kristensen (1986). The time interval between two decisions is called a stage, and since decisions are made every fourth week, all stages are of equal length. The possible decisions are to replace the cow or to keep it at least 4 weeks. As in similar models it is assumed that a replacement heifer is always available. A combination of values of genetic class, milk yield of previous lactation, milk yield of present lactation, calving interval, lactation number, and stage of lactation defines a state in the model. Apart from the states defined in this way, 3 additional states representing replacement, disease and infertility are included in the model in the same manner as described by Kristensen (1987*a*). The total number of states is 180 080 and only the formulation as a hierarchic Markov process makes it possible to obtain exact solutions to a model of that size. In Table 1 the characteristics of the model are summarized.

#### *Optimization and ranking*

A replacement policy is defined as a map assigning to each state of the model a decision ("keep" or "replace"). An optimal policy is a policy that maximizes a predefined objective function. In a situation without milk quotas the value of the objective function is

Table 1. *Characteristics of the dairy cow replacement model*

Number of stages in each lactation	11–18 <sup>a</sup>
Number of lactations considered	6
State variable of main process	
Genetic class <sup>b</sup>	5 classes
State variable of subprocesses	
Milk yield <sup>c</sup> of previous lactation	15 classes
Milk yield <sup>c</sup> of present lactation	15 classes
Length of calving interval	8 classes
Total number of states	180 080
Decisions	"keep" or "replace"

<sup>a</sup> Depending on the length of the calving interval.

<sup>b</sup> Defined from the relative breeding value of the father.

<sup>c</sup> Kg 4% milk adjusted as explained in the text.

the present value per cow (discounting criterion). In this case the optimization technique from Kristensen (1988) is applied. The ranking criterion is the future profitability defined as the difference in present value when a cow is kept and when it is replaced. If the optimal decision is to replace a cow the future profitability is negative.

Under milk quotas the values of the objective function is the long-run average net revenue per kg milk produced (average criterion). The optimization technique used in this case is briefly described in the appendix. Under these circumstances, the cows are ranked on the future profitability defined as the difference in relative value when the cow is kept and when it is replaced. The relative value in turn is equal to the economic value of a cow in a specific state measured as the deviation from a fixed basis.

In neither case the question of optimal herd size is answered. Under the discounting criterion the optimization assures that the net revenue per cow is as large as possible, and under the average criterion it is assured that each cow produces milk as cheaply as possible. If adjustment of herd size is necessary the ranking of cows is used in both cases.

#### *Technical and economical conditions*

The two criteria of optimality were compared under the production level, prices and interest rate appearing in Table 2. They are intended to refer to the situation of a typical Danish dairy herd. The net revenue is calculated as revenues from milk, calves and culled cows minus costs of feeds and heifers for replacement.

## RESULTS

### *Optimal replacement policies*

In Table 3 the optimal policies under both criteria are characterized by a number of key figures. As it appears the culling is much less intensive under the average criterion. The average time from first calving to replacement is 1.7 years under the discounting criterion and 2.6 years under the average criterion. As a result of the higher replacement rate under the discounting criterion the number of calvings per cow per year is 0.15 higher than under the average criterion. Also the milk yield per cow per year is largest under the discounting criterion. As expected the net revenue per kg of milk is largest under the average criterion since this is the objective which is maximized under that criterion. The difference represents 2.5% in the net revenue. This figure is a direct measure of the costs of using the traditional discounting criterion in a situation with milk quotas. If the herd quota is 700 000

Table 2. *Technical and economic conditions*

Prices (Dkr)	
Milk (kg FCM)	2.40
Basic food (SFU <sup>a</sup> )	1.30
Food for milk production (SFU)	1.45
Calf	1 400.00
Heifer	9 000.00
Young cow until 2nd calving (kg live weight)	11.50
Older cow (kg live weight)	11.00
Interest rate <sup>b</sup> (%)	3
Herd level <sup>c</sup> (FCM)	5 800

<sup>a</sup> Scandinavian Feed Units.

<sup>b</sup> Only discounting criterion (corrected for inflation).

<sup>c</sup> Average milk yield of the first 40 weeks of a lactation. Adjusted to 1st lactation level.

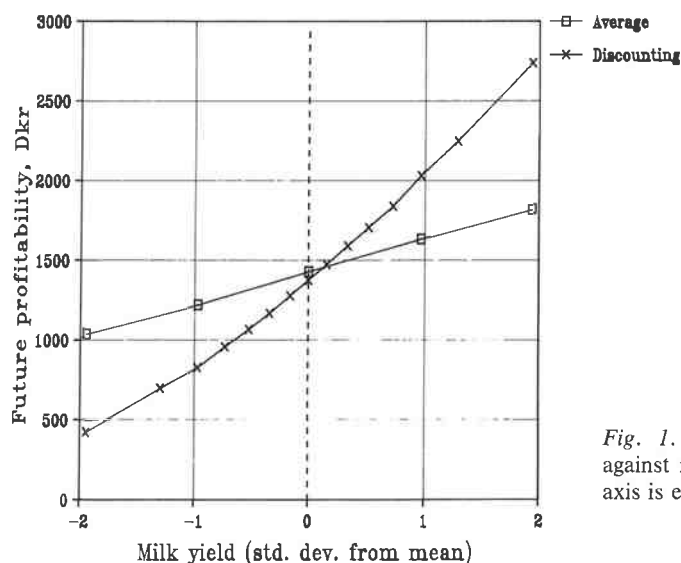


Fig. 1. Plot of future profitabilities against milk yield. One unit on the X-axis is equal to the standard deviation.

kg milk (approximately 100 cows) the difference in net income before tax equals 22 400 Dkr per year. Not surprisingly the net revenue per cow per year is largest under the discounting criterion. The difference of 220 Dkr is a direct measure of the costs of using the average criterion in a situation without milk quotas. In a herd with 100 cows this cost is 22 000 Dkr per year.

#### *Future profitability and ranking*

The ranking of states (cows) is based on the future profitability. In Table 4 the variation in future profitability (over states) under both criteria is described by percentiles. In the same manner the absolute difference in future profitability is shown. Differences in future profitabilities are not necessarily tantamount to differences in ranking, but may also reflect

Table 3. *Comparison of results from the optimization under both criteria*

The discounting criterion corresponds to a situation without milk quotas and the average criterion to a situation with quotas in effect. The results are under the optimal policies

	Criterion		
	Discounting	Average	Difference
Annual replacement rate (per cent)	58	38	20
Kg 4% milk per cow per year	7 333	6 989	344
Calvings per cow per year	1.29	1.14	0.15
Daily milk yield (first 24 weeks), kg 4% milk			
1st lactation	22.4	21.4	1.0
2nd lactation	26.9	25.3	1.6
3rd lactation	29.6	27.8	1.8
4th lactation	30.8	29.3	1.5
Length of calving intervals, days	373	381	-8
Net revenue per kg milk, Dkr	1.301	1.333	-0.032
Net revenue per cow per year, Dkr	9 538	9 318	220

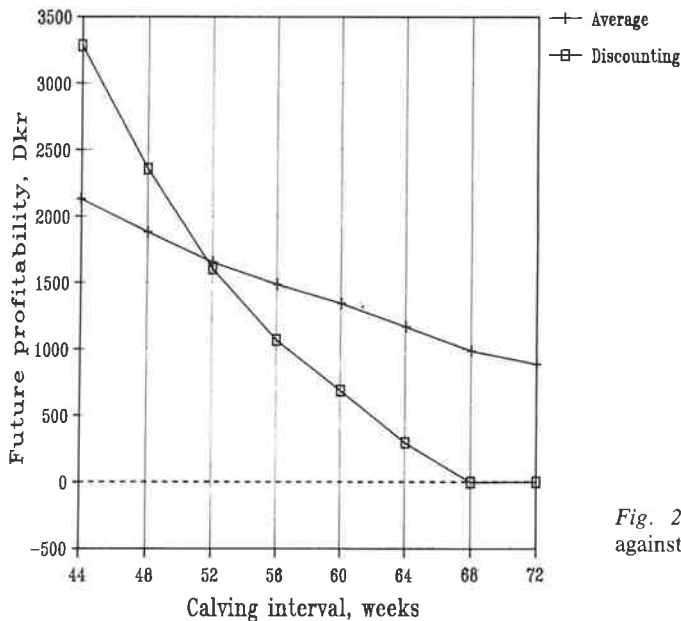


Fig. 2. Plot of future profitabilities against calving interval (weeks).

different levels and/or different standard deviations. In order to elucidate to what extent the differences in future profitability also reflect differences in ranking, the future profitabilities under the average criterion were standardized to have the same mean and standard deviation over states as under the discounting criterion. The following transformation was used:

$$p_a^* = ((p_a - m_a)/s_a) \times s_d + m_d, \quad (1)$$

where  $p_a^*$  is the transformed future profitability and  $p_a$  is the original value under the average criterion. The parameters  $m_a$ ,  $m_d$ ,  $s_a$  and  $s_d$  are the means and standard deviations over states under the average and discounting criterion respectively. The transformation (1) does *not* affect the ranking of states. If  $p_d$  (the future profitability under the discounting criterion) and  $p_a^*$  have different values in a state, it is a direct measure of different ranking under the two criteria. The percentiles of  $p_d - p_a^*$  are shown in the last column of Table 4.

In order to compare the variation in future profitabilities under the two criteria some plots showing the average future profitabilities for each level of state variables were made. In Fig. 1 the future profitabilities are plotted against milk yield expressed as the deviations from mean measured in units of standard deviation. Fig. 2 shows the future profitabilities 40 weeks after calving as functions of the calving interval.

## DISCUSSION

The less intensive culling under the average criterion is a consequence of the lower variability which is clear in Figs. 1 and 2. To understand the different slopes of the curves in Fig. 1 we shall consider the fundamental conditions of production in the two situations. When the limiting restriction is herd size (no quota) the only way of increasing net revenue is by improving the production efficiency of the individual cow. Since a way of obtaining a high production efficiency is a high milk yield this property will be given large weight. On the other hand, if quotas are in effect, we can choose between many cows having low milk yield and few cows having high milk yield. There is not the same need for exploiting each cow as intensively as when herd size is the limiting restriction. Therefore the advantage of

Table 4. Comparison of selected percentiles of future profitabilities and absolute differences in future profitabilities of states under the discounting and the average criterion (Dkr)

Percentiles	Future profitability		Absolute differences	
	Disc. $p_d$	Aver. $p_a$	Future profitability $ p_d - p_a $	Standardized future profitability $ p_d - p_a^* ^a$
100	6 801	3 660	3 411	2 922
90	3 225	2 170	1 372	1 220
75	2 398	1 864	1 034	878
50	1 358	1 482	644	503
25	297	1 042	312	228
10	-226	570	129	91
0	-909	-235	0	0

<sup>a</sup>  $p_a^* = ((p_d - m_d)/s_d) \times s_a + m_a$ , where  $m_a$ ,  $m_d$ ,  $s_a$  and  $s_d$  represent the mean and the standard deviation of future profitabilities over states under the average and discounting criterion respectively.

high milk yield is smaller. As we see from Table 3, the result is lower milk yield per cow under the average criterion.

In Fig. 2 the smaller variability under the average criterion can be explained by arguments similar to those given concerning milk yield. Also in this case the resulting larger calving intervals under the average criterion are confirmed by Table 3.

If we turn to Table 4 we find that the difference in future profitability of a state under the two criteria may be as large as 3 411 Dkr, and in 25 % of all states the difference exceeds 1 034 Dkr. Since the future profitability is an important source of information of a dairy farmer, the differences tell that it is of vital importance that the correct criterion is used. The last column in Table 4 expresses the economic significance of different ranking and it is obvious, that also the ranking of cows in a herd must be based on the correct criterion.

Based on the results we are able to conclude that the two criteria give rise to fairly different replacement policies. Because of a smaller variability in profitability of cows the culling is much less intensive under milk quotas. Also the differences in future profitability and the ranking of states are considerable. It is therefore important that the optimal replacement policy under milk quotas is calculated using the average criterion.

All results are based on the assumption that the presence (or absence) of the milk quota is expected to be permanent. If the quota is expected to be only temporary the results may be affected to an extent depending on the duration of the quota situation. If it is in effect during the entire expected lifetime of a cow, all results still hold because the differences in future profitabilities only arise from the remaining lifetimes of the cows in question (when a replacement takes place, the system is "reset"). If the quota is abolished already after one or two years, the situation is much more complicated, and no calculations have been performed to quantify the effect on the results under such circumstances.

## APPENDIX

### *Maximization of net revenue per kg milk using a hierarchic Markov process*

In the following only the optimization algorithm is given. For a deduction reference is made to Kristensen (1987b). For a general discussion of hierarchic Markov processes

reference is made to Kristensen (1988) where also the algorithm of the discounting criterion is given.

Let  $r_i^k(n)$  and  $m_i^k(n)$  be the net revenue and milk yield respectively of state  $i$  at stage  $n$  (a stage is a time interval) when the decision  $k$  is made ( $k = \text{"replace" or "keep"}$ ) in a subprocess. The transition probability from state  $i$  at stage  $n$  to state  $j$  at stage  $n+1$  under the decision  $k$  is denoted  $p_{ij}^k(n)$  in a subprocess. A set of decisions of all states at all stages in all subprocesses is called a policy and it is denoted  $S$ . Finally the net revenue per kg milk under a certain policy  $S$  is denoted  $g^S$ .

The total time of a subprocess is equal to the herd life of one cow in the model. The expected revenue  $R_l^S$  and milk yield  $M_l^S$  of an entire subprocess  $l$  (a cow) under a policy  $S$  are calculated recursively using

$$R_i^S(n) = r_i^k(n) + \sum_j p_{ij}^k(n) R_j^S(n+1),$$

where  $k$  is the decision defined by the policy  $S$ . The  $R_l^S$  is defined as

$$R_l^S = \sum_i p_i(0) R_i^S(1),$$

where  $p_1(0), \dots, p_I(0)$  are the probabilities of the states at the first stage of the  $l$ th subprocess. The value of  $M_l^S$  is calculated similarly. In the model there are 5 subprocesses each having a set of parameters  $r_i^k(n)$ ,  $m_i^k(n)$  etc. Each subprocess represents a genetic class. Finally  $q_l$  is the probability of the  $l$ th genetic class. We are now able to give the optimization algorithm as an iteration cycle in three steps:

1) Chose an arbitrary policy  $s$ . Go to 2.

2) Solve the following set of 5+1 linear simultaneous equations for  $g^S$  and  $h_1^S, \dots, h_5^S$ :

$$g^S M_l^S + h_l^S = R_l^S + \sum_{k=1}^5 q_k h_k^S, \quad l = 1, \dots, 5$$

$$h_5^S = 0.$$

Go to 3.

3) For each state  $i$  of each stage  $n$  of each subprocess  $l$ , find the decision  $k'$  that maximizes

$$r_i^{k'}(n) - m_i^{k'}(n) g^S, \quad n = N$$

$$r_i^{k'}(n) - m_i^{k'}(n) g^S + \sum_j p_{ij}^{k'} u_j(n+1), \quad n = N-1, \dots, 1,$$

and denote the maximum value of the expression  $u_i(n)$ . The parameters mentioned refer to the  $l$ th subprocess. The set of decisions  $k'$  defines a new policy  $S'$ . If  $S' = S$  then stop, since an optimal policy is found. Otherwise redefine  $S$  according to the new policy (i.e. put  $S = S'$ ) and go back to 2.

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# Bayesian updating in hierarchic Markov processes applied to the animal replacement problem\*

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## *Summary*

*The observed level of milk yield of a dairy cow or the litter size of a sow is only partially the result of a permanent characteristic of the animal; temporary effects are also involved. Thus, we face a problem concerning the proper definition and measurement of the traits in order to give the best possible prediction of the future revenues from an animal considered for replacement. A trait model describing the underlying effects is built into a model combining a Bayesian approach with a hierarchic Markov process in order to be able to calculate optimal replacement policies under various conditions.*

*Keywords: replacement, animal, Bayesian updating, Markov decision programming.*

## **1. Introduction**

In any production based on the operation of an asset of significant value, the determination of an optimal lifetime of the asset is important in order to maximise the profit from the production, which in this paper is assumed to be the overall objective of the manager. The considerations are relevant no matter whether the asset is a dairy cow, a farm building or some kind of industrial equipment, but the way of solving the problem may vary considerably, depending on the individual situation. Most often the asset will be replaced by a new asset of the same or at least a similar kind. In that case the present asset is only a link in a chain of assets. Then we have

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to consider what kind of objective function to use in order to meet the overall objective of profit maximisation.

In some cases it is relevant to maximise the total profit of the asset during the entire lifetime. That applies, for instance, when the asset itself is a scarce resource. A case in point is the determination of the optimal age at slaughter of fattening bulls in a dairy herd, where the main activity is milk production. If the dairy farmer does not buy bull calves at the market (for instance because of infection risk) he will only have the bull calves provided by the dairy cows of the herd. If the housing capacity is sufficient, the bull calves are a scarce resource, and the total profit is maximised if the net returns per animal during its whole lifetime are maximised. We shall refer to this situation as the *single asset* situation.

Another situation is when new assets are permanently available at the market. In that case the total profit is *not* maximised by maximising total net returns per asset. A more relevant criterion is here the maximisation of either average net returns over time or the total discounted net returns (i.e. the *present value*) of the entire chain of assets. In both cases the time horizon may be finite or infinite, whichever is relevant. An infinite horizon is just an abstraction indicating that the time of termination (the last link of the chain) is unknown, but at least 'far' ahead. We shall refer to this situation as the *asset chain* situation.

Finally, we shall consider a situation where a restriction is imposed on the production. It may be in the form of either a production quota or a limited supply of an input factor. In that case the total profit is maximised by the maximisation of average net returns per product or factor unit. We shall refer to this situation as the *quota* situation.

A more formal discussion of possible objective functions is given by Kristensen (1992a). The choice of objective function depends only on the conditions of production. It does not matter what kind of asset we are dealing with. If we turn to the *method* used in the maximisation of the objective function, it will depend very much on the nature of the asset. The classical replacement theory developed by pioneers like Preinrich (1940) and Terborgh (1949) typically assumes that all functions and parameters describing the problem are completely known in advance, and that no random variation is involved. The implicit items considered comprise machinery and other kinds of industrial equipment. The functions and parameters representing the problem are used for deduction of *general* replacement rules based on variants of the marginal net revenue approach.

As opposed to machinery or industrial equipment, the present study is part of a larger research project dealing with the *animal* replacement problem in agricultural production. A relevant question to ask is, therefore, in what way the animal problem differs from the general set-up. Based on a study by Ben-Ari et al. (1983) the main difficulties of the animal replacement problem may be summarised as:

- (1) *Uniformity*. The traits of an animal are difficult to define and measure.
- (2) *Variability*. The random variation of each trait is relatively large.
- (3) *Reproductive cycle*. The production of for instance cows and sows is cyclic. It has to be decided *in which* cycle to replace as well as *when* to replace inside a cycle.
- (4) *Herd restraints*. Animal production is performed in herds. There may be restraints that apply to the herd as a whole and not to the individual animal. Examples are a limited supply of heifers or gilts, limited housing capacity or a milk quota.

Because of the variability, Markov decision programming had already been applied to the dairy cow replacement problem by Giaever (1966). In an evaluation of techniques van Arendonk (1984) concluded that in dairy cow replacement this method should be used in preference to the marginal net revenue approach. Also, the method directly solves the problems caused by the reproductive cycle as shown by Kristensen and Østergaard (1982) as well as van Arendonk (1985b). The only problem concerning variability and cyclic production is that in order to cover the variability in traits, the state variables (traits) have to be represented by many levels, and to deal with the cyclic production, a state variable representing the stage of the cycle has to be included. Both aspects contribute significantly to an explosive growth of the state space. Therefore, we face a *dimensionality* problem. Though all necessary conditions of a Markov decision process are met, the solution is prohibitive in practice even on modern computers.

The problem of herd restraints is important. In dairy cattle two particular restraints should be considered. One is a limited supply of heifers when the dairy farmer only uses home-grown heifers as replacements. In that case a simple comparison of the animal in production with a replacement is no longer valid. Instead we face a much more difficult problem of choosing the optimal composition of animals from the available population of cows in production and heifers for replacement. The other restraint is the milk quota which is imposed on all dairy herds of the EC.

The overall objective of the animal replacement research project is to adapt the Markov decision programming techniques in order to be able to cope with the problem in a satisfactory way. The problems to be solved (totally or partially) have been identified as the dimensionality problem, herd restraints and uniformity. In order to circumvent the dimensionality problem, a new notion of a hierarchic Markov process was introduced by Kristensen (1988) and applied to the dairy cow replacement problem by Kristensen (1987, 1989). The technique may be applied in the single asset situation, the asset chain situation and the quota situation.

Both herd restraints mentioned above have been studied in the project. The milk quota restraint was discussed by Kristensen (1989) based on a technique described in Kristensen (1991) of maximising average net returns per kg milk produced. A limited supply of replacement heifers was discussed

by Ben-Ari and Gal (1986), who developed a technique called parameter iteration. The method was modified and further developed by Kristensen (1992b). In both studies on the latter restraint, a herd model was based on an underlying single animal model, which in the study by Ben-Ari and Gal (1986) was an ordinary Markov decision process and in the study by Kristensen (1992b) was a hierarchic Markov process.

Only a solution of the uniformity problem concerning the definition and measurement of traits remains to complete the project. The solution of that problem is the objective of the present study. In any replacement problem a good prediction of the future net revenues (or *rewards* as they are called in Markov decision programming) is essential. This prediction is based on the observed traits of the animal, but since the traits are subject to random variation, we do not know to what extent the observed value represents a permanent characteristic of the animal or just a temporary fluctuation. In order to be able to give the best possible prediction of future rewards, a method has been designed which combines the ability of knowledge updating known from causal probabilistic nets, as described for example by Pearl (1988), and hierarchic Markov processes. In order to keep the presentation simple we shall only consider examples where the animal in production is compared to a standard replacement. However, the technique may just as well be applied in a herd model under some restraint as discussed above.

For an empirical application of the updating technique in a sow replacement study, reference is made to Jørgensen (1992).

## 2. A model describing a trait of an animal

In this section we shall describe a general model of an animal trait to be used in replacement studies. The model will form the basis of the further considerations of the paper.

Assume that the *state* of an animal is observed at regular intervals called *stages*. The state is defined by the values of a number of *state variables* each representing a trait of the animal. We assume that one of the traits ( $Y_n$ ) is described by the following relation at stage  $n$ :

$$Y_n = m(\cdot) + X + e_n, \quad n = 0, \dots, N, \quad (1)$$

where  $m$  is a known function expressing the expected value of  $Y_n$  under the circumstances in question. The circumstances are represented by the arguments of  $m$  which may be the value of other state variables (e.g. the age of the animal, the season, etc.) and/or the average herd level concerning the trait. The trait itself may, for instance, be the milk yield of a dairy cow or the litter size of a sow. The symbol  $X$  is the combined effect of genetic level and permanent environment. We assume  $X$  to be normally distributed with the expected value zero and a certain variance  $\sigma_x^2$  representing the variation

across the population. The symbol  $e_n$  represents the random variation caused by temporary environmental effects. We assume that  $e_n$  is normally distributed with the expected value zero and a certain variance  $\sigma_e^2$ . The random variables  $X$  and  $e_n$  are assumed to be independent. Furthermore, the variables  $e_1, \dots, e_N$  of a particular animal are assumed to be independent of the corresponding variables of the other animals of the herd. Thus all systematic effects influencing all animals of the herd (e.g. seasonal effects) are assumed to be included in the function  $m(\cdot)$ . The sum  $I_n = X + e_n$  forms the value of a state variable.

The relation over time of  $e_1, \dots, e_N$  is described by a first order autoregressive process, i.e.

$$e_n = ae_{n-1} + \varepsilon_n, \quad n = 1, \dots, N, \quad (2)$$

where  $0 < a < 1$  and  $\varepsilon_1, \dots, \varepsilon_N$  are independent and normally distributed with the expected value zero and the variance  $(1 - a^2)\sigma_e^2$ . Furthermore,  $\varepsilon_n$  is assumed to be independent of  $e_{n-1}$  and  $X$  for  $n = 1, \dots, N$ .

As appears from equations (1) and (2), the permanent effect  $X$  varies only between animals, whereas the temporary random effect  $e_n$  varies over time for the same animal. It is obvious that the value of the permanent effect is very important in the decision of which animals to keep in the *long run* (e.g. whether a cow should be kept for an additional lactation) and that the current value of the random effect  $e_n$  is important in the decision concerning the optimal replacement time in the *short run* (e.g. when to replace a cow inside a lactation). Thus, if the value of  $X$  is high, we would probably ignore a low current value of  $e_n$  which just represents a temporary crisis. On the other hand, a sufficiently high current value of  $e_n$  might lead to postponed replacement of an animal of low permanent value  $X$ .

These fundamental observations illustrate that the differentiation of variation between animals and over time for the same animal is important for the replacement decision, because it directly influences our expectations concerning the future net revenues from the animal. The only problem is that neither the permanent effect  $X$  nor the current random effect  $e_n$  are directly observable. What we observe are the resulting numerical values  $Y_1, \dots, Y_N$  of the trait in question, but since the systematic effect  $m$  is assumed to be known, this is equivalent to the sums  $I_1, \dots, I_N$ , where  $I_n = X + e_n$ . On the other hand we may have a prior belief in  $X$  based on an estimated variance among animals and possible observations concerning the animal of characteristics correlated with  $X$ . Further, as observations of the sums  $I_n$  are gathered they will increase our knowledge of  $X$ . If, for instance, all sums are relatively large for an animal, it implicitly indicates a high value of  $X$  and vice versa.

The trait model (1) and (2) may easily be extended to cover several traits, each being influenced by several unobservable effects. Assume for instance that  $Y_{1n}$  and  $Y_{2n}$  are the milk yield and weight of a dairy cow at stage  $n$  and

that each trait is influenced by two unobservable effects. In matrix notation we may express the relations as follows:

$$Y_n = \begin{pmatrix} m_1(\cdot) \\ m_2(\cdot) \end{pmatrix} + \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} 1 \\ c_n \end{pmatrix} + \begin{pmatrix} e_{1n} \\ e_{2n} \end{pmatrix}, \quad (3)$$

where  $Y_n = (Y_{1n}, Y_{2n})'$ , and

$$\begin{pmatrix} e_{1n} \\ e_{2n} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} e_{1,n-1} \\ e_{2,n-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1n} \\ \varepsilon_{2n} \end{pmatrix}. \quad (4)$$

If, at a previous stage, the cow has been suffering from a specific disease (e.g. mastitis) that permanently influences the milk yield and weight, the value of  $c_n$  is  $-1$ . Otherwise it is zero. Equation (3) expresses that the observed milk yield and the weight of the cow are determined partly by permanent animal-specific effects ( $X_{11}$  and  $X_{21}$ ) as in the single trait model (1), partly by possible permanent negative effects of a previous disease ( $X_{12}$  and  $X_{22}$ ) and finally by temporary random effects ( $e_{1n}$  and  $e_{2n}$ ). The variables  $X_{11}$ ,  $X_{12}$ ,  $X_{21}$  and  $X_{22}$  may be mutually correlated, and  $e_{1n}$ ,  $e_{2n}$  may be correlated, whereas  $(X_{11}, X_{12}, X_{21}, X_{22})'$  and  $(e_{1n}, e_{2n})'$  are assumed to be independent.

In the following, only the single-trait model of equations (1) and (2) will be discussed, but all results may be directly extended to cover the multi-trait model of equations (3) and (4). For the derivation of the extended results, a Kalman filter approach as described by, for example, Harrison and Stevens (1976) is a relevant tool.

We refer to  $X$  in equation (1) as the *basic state* of the animal and to the sum  $I_n$  as the *current state*. Thus the current state is directly observable and therefore known at any stage, whereas the basic state is unknown. In accordance with common practice in dynamic programming we shall consider both kinds of states to be discrete, i.e. only a finite number of levels are considered for each kind. Both  $X$  and  $I_n$  are random variables, which will be referred to in upper-case letters. Transformed realisations of  $X$  and  $I_n$ , on the other hand, will be denoted by lower-case letters  $x \in \Omega_X$  and  $i \in \Omega_I$ , respectively, where  $\Omega_X$  and  $\Omega_I$  are finite sets. In other words, if the basic state is  $x$ , it means that  $X$  is in the interval  $]x^-; x^+]$  defined as  $\{y | x^- < y \leq x^+\}$ , where  $x^-$  and  $x^+$  are the lower and higher limit, respectively, of the  $x$ 'th level of  $X$ . In addition to the states defined by levels of  $I_n$ , the state space  $\Omega_I$  includes a *replacement state* representing a situation where the animal has been culled.

From the assumptions made concerning the normal distribution of  $X$ , we are able to calculate the prior probability  $p_x(0)$  of any basic state  $x$ . At any stage  $n$  we may select an *action*  $d \in \{1, 2\}$  that influences the system. We shall interpret  $d = 1$  as 'keep' and  $d = 2$  as 'replace'. For given basic state  $x$  and current state  $i$  at stage  $n$ , we know the conditional probability  $p_{xij}^d(n)$  of the

current state to be  $j$  at stage  $n + 1$  if action  $d$  is taken. For  $d = 1$  we have approximately:

$$\begin{aligned}
 p_{xij}^1(n) &= P(j \in ]j^-; j^+ ] | x \in ]x^-, x^+ ] \wedge i \in ]i^-; i^+ ]) \\
 &\approx P(x^m + e_{n+1} \in ]j^-; j^+ ] | X = x^m \wedge I = i^m) \\
 &= P(e_{n+1} \in ]j^- - x^m; j^+ - x^m ] | X = x^m \wedge e_n = i^m - x^m) \\
 &= \Phi((j^+ - x^m - a(i^m - x^m))/(1 - a^2)^{1/2} \sigma_e) \\
 &\quad - \Phi((j^- - x^m - a(i^m - x^m))/(1 - a^2)^{1/2} \sigma_e),
 \end{aligned} \tag{5}$$

where  $\Phi$  is the distribution function of the standard normal distribution. The symbol  $x^m$  denotes the conditional expectation  $E(X | X \in ]x^-; x^+ ])$ , and analogously for  $i^m$ . If  $d = 2$ , the process transfers to an absorbing replacement state with probability 1.

At stage 0, the marginal probability of a transition from current state  $i$  to  $j$  at stage 1 under the action  $d$  is calculated as

$$p_{ij}^d(0) = \sum_{x \in \Omega_x} p_{xij}^d(0) p_x(0). \tag{6}$$

At each stage the current state  $i$  is observed, each time increasing our knowledge of the basic state  $x$ . Our current belief at stage  $n$  concerning the basic state is represented by the probability distribution given by  $p_x(n)$ . If the current state is  $i$  and at stage  $n + 1$  we have observed a transition from state  $i$  to state  $j$  following the action  $d$ , we may use Bayes' theorem to update our belief concerning the basic state. The new probability distribution at stage  $n + 1$  is calculated as

$$p_x(n + 1) = p_x(n) p_{xij}^d(n) / p_{ij}^d(n), \quad x \in \Omega_x, \quad n = 0, \dots, N - 1. \tag{7}$$

If the current state  $i$  is observed at stage  $n$ , and the action  $d$  is taken, a *reward* depending on the basic state  $x$  is gained. This reward is denoted as  $r_{xi}^d(n)$ . We also assume that some kind of physical output  $m_{xi}^d(n)$  is produced during the stage. In a replacement model, the reward is usually defined as the net revenue, and the physical output may be defined as the amount of milk produced by a cow, the litter size of a sow, etc.

### 3. Causal probabilistic nets

A trait described as in section 2 may be modelled by a *causal probabilistic net* (sometimes also referred to as an *inference diagram* or a *Bayes belief net*) as shown in Figure 1. Using the terminology of Tatman and Shachter (1990), the elements of the net are *decision nodes* representing variables under the control of the decision maker, *chance nodes* corresponding to random variables or random events, *value nodes* together representing the arguments of

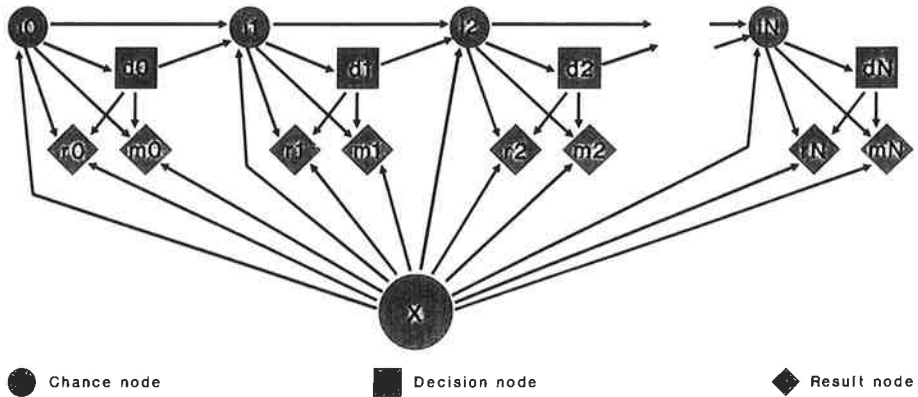


Figure 1. The animal model described as a causal probabilistic net

$X$  = basic state  
 $i_n$  = observable state at stage  $n$   
 $d_n$  = action at stage  $n$   
 $r_n$  = reward at stage  $n$   
 $m_n$  = physical output at stage  $n$

the objective function of the model and, finally, *directed arcs* representing the causal relationships among the nodes. Arcs into a decision node indicate the information which will be known to the decision maker at the time of decision. Arcs into a chance node indicate which variables condition the probability distribution of the associated random variable. Arcs into a value node indicate which variables condition the associated expected value (arcs from a value node to another node are not allowed).

An advantage of causal probabilistic nets is that they provide a graphical modelling language very close to ordinary human reasoning, and at the same time they are mathematically well defined and, therefore, suitable for analyses derived from traditional probabilistic theory. The main idea is that the knowledge of unobservable nodes (state variables) is updated each time the value of any other node is observed. Thus, in the example, the model is learning by successive observations, and step by step knowledge concerning the value of  $X$  is increased. By assuming a decision policy describing which decision to make for given values of the chance nodes it is possible to calculate the expected value of the objective function under the policy. This value may be compared to the expected values under alternative policies, making possible the identification of an optimal (or at least a satisfactory) policy. During the last few years there has been extensive research into causal probabilistic nets, and many results have been obtained on how to collect and distribute evidence over the net (e.g. Pearl, 1988; Jensen et al., 1990).

Possible objective functions are the expected sum of all rewards under a policy (ignoring the physical output) or the expected sum of all discounted rewards under a policy (if the physical output is defined to be the stage length). Thus, the causal probabilistic net of Figure 1 may be used directly

for solving the problem described above, as long as the time horizon is restricted to the  $N$  stages. In other words, the causal probabilistic net directly solves the problem of determining the optimal lifetime of an animal in the single asset situation as defined in section 1. If, on the other hand, the system at the end of the  $N$  stages is replaced by a new system described in the same way (and a third system will ultimately replace the second one and so on) the causal probabilistic net approach will run into trouble, because all time steps explicitly have to be in the model. Thus, the method does not cover the asset chain and quota situations mentioned in section 1.

The infinite stage problem is dealt with appropriately by a hierarchic Markov process, but the problem of that method in relation to the current problem is that all states must be observable and all parameters must be known. The possibility of learning from the successive observations is not directly present. Therefore, a hybridisation of a hierarchic Markov process and a causal probabilistic net is desirable in order to cover the asset chain and quota situations as described in section 1.

#### 4. Hierarchic Markov processes

If we want to describe the system by a Markov decision process instead of a causal probabilistic net, we may take at least two different views. One possibility is to define it as a Markov decision process with unobservable states. Such processes are called *partially observable Markov decision processes*, and they have been discussed by Monahan (1982). Another view to take is to define the process in such a way that the state space is directly observable, but with unknown parameters. This kind of process is called an *adaptive Markov decision process*. It has been discussed in detail by Wessels (1968) and later by van Hee (1978).

However, in this paper we shall consider the system in the context of a hierarchic Markov process. A hierarchic Markov process is a series of finite stage Markov decision processes called *subprocesses* built together in one Markov decision process called the *main process*. The basic formulation, including an optimisation cycle for the discounting (present value) criterion, is given by Kristensen (1988). This criterion covers the asset chain situation. A similar cycle intended for the quota situation, using a criterion maximising the average rewards/output ratio, is described by Kristensen (1991). The hierarchic technique has been developed as a way of circumventing the dimensionality problem of Markov decision programming, i.e. that practical problems have a tendency to become very large and, therefore, difficult to analyse by the usual techniques as mentioned in section 1. In the description of hierarchic Markov processes it has been assumed that all parameters of the model are known and that all states in the subprocesses as well as in the main process are directly observable.

If, in the specification of traits in section 2, the basic state  $X$  had been directly observable, the formulation as a hierarchic Markov process would have been straightforward. In that case the state space of the main process would have been the finite set  $\Omega_X$ , and the state space of the subprocesses would be  $\Omega_i$ . The parameters of the  $x$ 'th subprocess would be  $p_{xij}^d(n)$ ,  $r_{xi}^d(n)$  and  $m_{xi}^d(n)$ . Finally, the  $x$ 'th element of the  $y$ 'th row of the main process transition matrix would be  $p_x(0) = \Phi(x^+/\sigma_x) - \Phi(x^-/\sigma_x)$ . Under these assumptions we would be able to determine an optimal solution for the asset chain situation as well as for the quota situation.

Since, however, the basic state  $x$  is unobservable, we may conclude that we have a hierarchic Markov process with *unobservable main state*. But at each stage we observe the state transition in the subprocess and use the observation for updating our knowledge of the state of the main process. At the end of the subprocess, however, the learning stops, because the knowledge of the old system cannot be used on the new one. In other words, the specific traits of the present animal will not improve our ability to predict the future revenues from the replacement.

In the following we shall describe how the updating of knowledge may be incorporated into the hierarchic process. First, we should notice that, for a given transition from state  $i$  to  $j$  in a subprocess, the new probability distribution of the main stage  $x$  is uniquely defined according to equation (7). Therefore, the imperfect knowledge of the main state does not add any further random elements to the transitions of the subprocesses. Next, we should consider whether we know that the distribution of  $x$  always belongs to a certain class of distributions so that it may be sufficiently described by one or a few parameters. In that case we may replace  $p_1(n), \dots, p_v(n)$ , where  $v$  is the number of elements in  $\Omega_X$ , by these few parameters without losing any information. The prior distribution of  $X$  is normal with known mean and variance. In the following we shall investigate the posterior distribution after observations of current states in the subprocesses.

Having observed state  $i$  through the sum  $I_n = X + e_n$  at stage  $n$ , we are ready to observe the state  $j$  defined by the sum  $I_{n+1} = X + e_{n+1}$  at the next stage. Recalling that  $X$  has a *fixed* (but unknown) value, we find that the distribution of  $I_{n+1}$  is normal, having the expected value

$$E(I_{n+1} | X, I_n) = E(X + a(I_n - X) + \varepsilon_{n+1} | X, I_n) = X + a(I_n - X) \quad (8)$$

and variance

$$V(I_{n+1} | X, I_n) = V(X + a(I_n - X) + \varepsilon_{n+1} | X, I_n) = (1 - a^2)\sigma_e^2. \quad (9)$$

From equations (8) and (9) we observe that  $I_{n+1}$  has unknown mean but known variance. Our prior knowledge of the mean is that it is normally distributed with the expected value

$$\mu_{n+1} = E(X + a(I_n - X)) = (1 - a)E_n(X) + aI_n \quad (10)$$

and the variance

$$\sigma_{n+1}^2 = V(X + a(I_n - X)) = (1 - a)^2 V_n(X), \quad (11)$$

where the index  $n$  on the expectation and variance of  $X$  refer to the distribution of  $X$  at stage  $n$ .

Having taken the observation of  $I_{n+1}$  we may update the distribution of the mean according to the following equations taken from DeGroot (1970):

$$\mu'_{n+1} = (\mu_{n+1} \sigma_{n+1}^{-2} + I_{n+1} (1 - a^2)^{-1} \sigma_e^{-2}) / (\sigma_{n+1}^{-2} + (1 - a^2)^{-1} \sigma_e^{-2}) \quad (12)$$

and

$$\sigma_{n+1}'^2 = \sigma_{n+1}^2 (1 - a^2) \sigma_e^2 / (\sigma_{n+1}^2 + (1 - a^2) \sigma_e^2). \quad (13)$$

Furthermore, the posterior distribution of  $I_{n+1}$  is still normal according to DeGroot (1970). By combining equations (10) and (11) with (12) and (13) we are able to calculate the parameters of the new distribution of  $X$  at stage  $n + 1$ :

$$\begin{aligned} E_{n+1}(X) &= (\mu'_{n+1} - aI_n) / (1 - a) = (1 - a^2) \sigma_e^2 E_n(x) \\ &\quad + (I_{n+1} - aI_n) (1 - a) V_n(x) / ((1 - a^2) \sigma_e^2 + (1 - a)^2 V_n(x)) \end{aligned} \quad (14)$$

and

$$\begin{aligned} V_{n+1}(x) &= \sigma_{n+1}'^2 / (1 - a)^2 \\ &= (1 - a^2) \sigma_e^2 V_n(x) / ((1 - a)^2 V_n(x) + (1 - a^2) \sigma_e^2). \end{aligned} \quad (15)$$

Under the assumptions made we find that if the prior distribution of  $X$  is normal, it will remain normal at all stages. Only the expectation and variance change over stages, and furthermore the change in variance does not depend on the observed value of  $I_{n+1}$ ! For given prior variance  $V_0(x) = \sigma_x^2$  we are able to calculate the variance at all future stages in advance according to the recurrent equation (15). If instead of the variance we consider the reciprocal value (sometimes referred to as the *precision*), we easily have

$$1/V_n(x) = n(1 - a)^2 / (1 - a^2) \sigma_e^2 + 1/V_0(x) \quad (16)$$

showing that the precision increases linearly with  $n$ . Thus the variances should be considered as known in advance, and only the changes in expected value depend on the observations made. It will not be necessary to keep the probabilities  $p_x(n)$ . It is sufficient to keep the expected value of  $X$ , and the probabilities  $p_x(n)$  may at any stage be reproduced by the relation

$$p_x(n) = \Phi((x^+ - E_n(x)) / (V_n(x))^{1/2}) - \Phi((x^- - E_n(x)) / (V_n(x))^{1/2}). \quad (17)$$

As a consequence of this finding, we now redefine the state spaces of the hierarchic Markov process so that the state space of the main process holds

only one element, and the state space of the only possible subprocess becomes  $\Omega_2 = \{\mu_1, \dots, \mu_m\} \times \Omega_I$ , where the set  $\{\mu_1, \dots, \mu_m\}$  represents alternative values of the expectation of  $X$ . In the following  $\mu(i)$  will denote the expected value belonging to state  $i \in \Omega_2$ .

Put  $E_n(x) = \mu(i)$  and  $E_{n+1}(x) = \mu(j)$  for any  $i, j \in \Omega_2$ . If  $E_n(x)$  and  $E_{n+1}(x)$  satisfy equation (14), the transition probabilities of the subprocess are calculated as

$$p_{ij}^1(n) = \sum_x p_{xij}^1(n) p_x(n) \quad (18)$$

and otherwise

$$p_{ij}^1(n) = 0. \quad (19)$$

The expected production and reward given stage, state and action are calculated as

$$m_i^d(n) = \sum_x m_{xi}^d(n) p_x(n), \quad (20)$$

and

$$r_i^d(n) = \sum_x r_{xi}^d(n) p_x(n). \quad (21)$$

We have now arrived at an ordinary hierarchic Markov process that may be solved by usual methods as described by Kristensen (1988, 1991). Thus we are able to solve the asset chain situation as well as the quota situation.

## 5. Benefits from updating: a numerical example

In order to illustrate the benefits of updating, we shall consider a numerical example. Suppose that the expected production of an animal decreases linearly with age from an initial level at stage 0 according to the following relation:

$$m(n) = c_1 - c_2 n. \quad (22)$$

The observed total production of an animal during stage  $n$  is calculated as  $Y_n$  in equation (1), and we define the physical output as  $m_{xi}^d(n) = Y_n$  for  $d \in \{1, 2\}$ , since a replacement is assumed to take place (and to be decided) at the end of a stage. In the replacement state, the physical output is zero. The reward gained at stage  $n$  is defined as

$$r_{xi}^d(n) = c_3 m_{xi}^d(n) - c_4(n) + c_5^d(n), \quad (23)$$

where  $c_3$  is the unit price of the product,  $c_4(0)$  is the price of a new animal for replacement and  $c_4(n) = 0$  for  $n > 0$ ,  $c_5^1(n) = 0$  for all  $n$ , and  $c_5^2(n)$  is the

value of an animal being replaced at stage  $n$ . In the replacement state, however, the reward is zero. We shall assume the value of an animal to decrease linearly over stages from an initial carcass value  $c_6$  at stage 0 according to the relation

$$c_5^2(n) = c_6 - c_7 n. \quad (24)$$

A set of numerical values were chosen for  $\sigma_x$ ,  $\sigma_e$ ,  $a$ ,  $N$  and the  $c$ -constants of equations (22) to (24). The selected values are summarised in Table 1. For  $X$ , the nine levels  $]-\infty; -3.5]$ ,  $]-3.5; -2.5]$ , ...,  $]3.5; \infty[$  were distinguished. The levels are referred to as 1, ..., 9, respectively. For the sums  $I_n = X + e_n$ , the 13 levels  $]-\infty; -5.5]$ ,  $]-5.5; -4.5]$ , ...,  $]5.5; \infty[$ , referred to as 1, ..., 13, were considered. For the current *expected* value of  $X$ , the same levels are used as for  $X$ . Thus the total number of states in the subprocess becomes  $9 \times 13 + 1 = 118$  (the last state added is the replacement state).

In order to be able to evaluate the benefit of updating, two alternative hierarchic models were formulated. In one model it was assumed that  $X$  was directly observable. In that case the nine levels of  $X$  were defined as states of the main process, and the 13 levels of  $I_n$  plus the replacement state were defined as 14 states in the subprocesses.

The second alternative represents a situation where  $X$  is not observable, and no updating of the belief in  $X$  is performed. In other words, the prior distribution of  $X$  is used during the whole lifetime of the animal. The same hierarchic design was used as in the updating situation, but in equations (18), (20) and (21) the *initial* state distribution  $p_1(0), \dots, p_9(0)$  of  $x$  was used at all stages instead of  $p_1(n), \dots, p_9(n)$ . In all three models, optimal policies under the discounting criterion referring to the asset chain situation were calculated using the optimisation cycle of Kristensen (1988), and the economic results were measured by the present value of the entire infinite process calculated just before purchase of a new animal. The results are compared in Table 2.

Table 1. *Selected values for the parameters of the numerical example in section 5*

Parameter description	Symbol	Value
Standard deviation of basic state $X$	$\sigma_x$	2
Standard deviation of temporary effect $e_n$	$\sigma_e$	2
Autoregression coefficient	$a$	0.5
Maximum age (stages) of an animal	$N$	10
Expected production of an animal at stage 0	$c_1$	20
Expected reduction per stage in production	$c_2$	0.5
Unit price of product	$c_3$	10
Price of new animal for replacement	$c_4(0)$	200
Carcass value of an animal at stage 0	$c_6$	120
Reduction per stage in carcass value	$c_7$	4
Discount factor per stage		0.95

Table 2. *Present values under optimal policies in three alternative situations, representing different levels of knowledge on the basic state  $X$* 

Level of knowledge	Present value	Relatively
Only prior knowledge available	3120	100
Updating of knowledge as observations are done	3435	110
Complete knowledge of the value of $X$	3449	111

As it appears, the updating of knowledge increases the economic result by 10% compared to a situation with no updating. Furthermore, the result under updating is very close to the result under complete knowledge. The autoregression coefficient  $a$  is a measure of the constancy of the random effect  $e_n$ . In the extreme case  $a = 0$ , the variation of  $e_n$  is just noise in the observation of  $X$ , whereas in the opposite situation with  $a = 1$ ,  $e_n$  will be constant over time, making  $X$  and  $e_n$  measure exactly the same, namely a permanent characteristic of the animal. In order to study the effect of  $a$  on the benefit of updating the value was varied from 0.1 in steps of 0.1 to a value very close to 1.

It appears from the results that there is practically no benefit of updating when  $a$  is close to 1, whereas the benefit is considerable for small values of  $a$ . The reason is that for values of  $a$  close to 1,  $X$  and  $e_n$  express almost the same, and in that case only the directly observable sum is of interest. Further, it appears from equation (16) that the precision of the belief concerning  $X$  increases only very little over stages when  $a$  is close to 1, since the increase per stage is proportional to the factor  $(1 - a)^2 / (1 - a^2)$  which decreases for  $a$  increasing towards 1. Therefore, the benefit from updating is very small for such high values of  $a$ .

In the consideration of the influence of  $a$  on the benefit of updating, two arguments lead in the same direction: (1) the precision increases only a little for values of  $a$  close to one, and (2) the economic significance of distinguishing  $X$  and  $e_n$  vanishes as  $a$  converges towards 1. We shall now consider the influence of combined values of the standard deviations  $\sigma_x$  and  $\sigma_e$ . Concerning this question we are less fortunate than when we considered the influence of  $a$ . On the one hand, we know from equation (16) that the increase in precision concerning the belief in  $X$  is small when  $\sigma_e$  is big. On the other hand, if  $\sigma_e$  is small compared to  $\sigma_x$ , the directly observed sum  $I_n = X + e_n$  expresses almost the same as  $X$ . Therefore, we must expect the benefit of updating to be relatively small, since we are told something that we (almost) knew in advance. Thus, we have conflicting views, and only the results may show us the true influence of  $\sigma_n$ .

It was found that for a fixed value of  $\sigma_x$  the benefit *increases* with  $\sigma_e$ . We may therefore conclude that even though the information acquired is rather vague for high values of  $\sigma_e$ , it is at least *new* and therefore more valuable

than for low values of  $\sigma_e$ . It also appeared that the benefit from updating increases even more with  $\sigma_x$ . This is not surprising, since a great variation in a variable automatically increases the economic value of information concerning the true level of the variable.

## 6. Discussion

The numerical example in section 5 has shown that the benefit of updating may be considerable. In order to evaluate the method more carefully, we shall now compare it to the methods typically used for modelling trait variations in replacement studies in the literature. In most cases the state variables have been defined from the directly observable variables  $I_1, \dots, I_N$ , but the variables have typically not been regarded as sums of underlying unobservable effects. Examples in dairy cattle are Giaever (1966), Smith (1971), McArthur (1973), Kristensen and Østergaard (1982), van Arendonk (1985a, 1985b) and Kristensen (1986, 1987). An example in sows is the work of Huirne et al. (1988).

All authors mentioned have been aware that part of the observed value of  $I_n$  is due to a permanent property of the animal, even though it has not been formulated directly as is done in equations (1) and (2). Without such a model, the ideal way to take the permanent effect into account is to use *all* previous observations  $I_1, \dots, I_n$  in the prediction of  $I_{n+1}$ . Thus all observed values should be kept as state variables in the model. Therefore, the size of the state space becomes prohibitive if an appropriate number of levels is defined for each of them.

The most common way of dealing with this problem in the literature is to assume that the last two or three observations ( $I_{n-2}, I_{n-1}, I_n$ ) are sufficient in the prediction of  $I_{n+1}$ . Thus we only have to keep two or three state variables instead of all  $n$  observations. This method was used in dairy cows by Smith (1971), van Arendonk (1985b) and Kristensen (1986, 1987), all keeping two observations of  $I_n$ . In sows, Huirne et al. (1988) used the same approach keeping three observations of litter size.

In the approach taken in this paper,  $I_n$  is assumed to be the sum of two (or more) underlying unobservable effects as defined in equation (1). Under these conditions, it is easily shown that the expected value and variance of  $I_{n+1}$  given  $I_1, \dots, I_n$  are calculated as

$$E(I_{n+1} | I_1, \dots, I_n) = (1 - a)E(X | I_1, \dots, I_n) + aI_n \quad (25)$$

and

$$V(I_{n+1} | I_1, \dots, I_n) = (1 - a)^2 V(X | I_1, \dots, I_n) + (1 - a^2)\sigma_e^2. \quad (26)$$

From equations (25) and (26) we are able to conclude that by keeping only the current expectation of  $X$  and the most recent observation  $I_n$ , the predic-

tion of  $I_{n+1}$  is exactly the same as if all previous values  $I_1, \dots, I_n$  were kept and used in the prediction! As shown in equation (16), the conditional variance of  $X$  is independent of the observations made and known in advance.

We are able to conclude that if the model represented by equations (1) and (2) is true, we only need two state variables in the model to obtain the same precision as if all previous observations of the trait in question were kept as state variables. Since the dimension of the model is the more limiting restriction in practical applications of Markov decision processes, this is an important contribution to the problem of reducing the state space without loss of precision. The generalisation of this conclusion to the multi-trait model sketched in section 2, equations (3) and (4), is that the number of necessary state variables equals the sum of directly observed traits and the number of unobservable permanent effects.

It must be emphasised that the multi-trait formulation in equations (3) and (4) is not necessary in all cases where several random traits are observed. If, in addition to  $Y_n$ , another trait  $Z_n$  is observed, and this trait only affects  $Y_n$  through the function  $m(\cdot)$  so that  $I_n$  and  $Z_n$  are independent there is no problem in treating such a case within the single-trait model. The transition probabilities concerning  $I_n$  described in this paper simply have to be multiplied by those of  $Z_n$ .

The definition of a directly observable trait as a sum of underlying unobservable effects provides a framework for prediction of the future revenues in an optimal way given the information available at the time of decision. Thus the conclusion of the present study is that the developed knowledge updating technique seems to be an appropriate solution to the uniformity problem of defining and measuring the traits of an animal considered for replacement in the asset chain situation as well as in the quota situation.

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## Optimal Replacement in the Dairy Herd: A Multi-component System\*

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### ABSTRACT

*The dairy herd is described as a multi-component system, where the components are the cows and heifers. The problem of finding an optimal replacement policy to the multi-component system is considered. The complication of the multi-component model is that, if the supply of heifers is limited (i.e. the dairy farmer uses only home-grown heifers), the replacement decision concerning a cow does not only depend on the state of that particular cow but also on the states of the other cows and heifers in the herd. Initially, it is demonstrated that the multi-component replacement problem may be formulated as an ordinary Markov decision process. Unfortunately, the model is far too large to be solved by any known methods. Therefore, an approximate method combining dynamic programming and stochastic simulation in the determination of a set of descriptive parameters is suggested. The parameters are used in the calculation of the multi-component replacement criterion for cows as well as for heifers. The method has been tested by extensive simulations under 100 different conditions concerning prices and average milk yield of the herd. It was concluded that, when the replacement costs (the price of a heifer minus the price of a calf and the carcass value of a cow) are small, the method improves the economic results considerably compared to the usual models, assuming an unlimited supply of heifers. The information concerning heifers, which is provided by the method, makes it relevant even in cases where the replacement costs are large. The basic idea of the study may be relevant in a more general range of problems involving replacement under some constraint.*

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## 1 INTRODUCTION

The aim of this study was to develop a method for finding an approximate solution for the optimal replacement policy in the dairy herd regarded as a multi-component system. In the literature, several studies deal with optimal dairy cow replacement regarded as a single-component system (i.e. only one cow is considered at a time, assuming an unlimited supply of heifers for replacement). A review of such studies is given by Arendonk (1984), and more recent examples are Arendonk and Dijkhuizen (1985), Arendonk (1986) and Kristensen (1987, 1989). The latter models are very detailed and, from a theoretical point of view, the single-component replacement problem in dairy herds may be regarded as having been solved to a satisfactory degree.

Most replacement studies in the literature deal with a single-component system (i.e. only one producing unit (component) is considered, assuming an unlimited supply of replacement units). In a multi-component system, however, several producing units (components) are considered simultaneously. If only the number of components in operation is limited, but the supply of replacements is unlimited, the problem is identical to that of the single-component model, because the decision to replace a component does not influence the possibility of replacing any of the other components. On the other hand, if the supply of replacements is limited, the replacement of one component will decrease the possibility of replacing others, because the number of replacements may not suffice. Therefore, the replacement decision concerning one component does not only depend on the state of that particular component but also on those of the other components of the system. Instead of a relatively simple comparison of two components (the one in operation and the replacement) the problem is now faced of choosing the optimal composition of components from the available population of the components in operation and the available replacements.

It is assumed that the interaction is due to technical and economic dependency so that each component is assumed stochastically independent of the others. If the dairy herd is regarded as a multi-component system, the system is the entire herd, whereas the components in operation are the individual cows and the replacements are the available heifers.

A special feature of the dairy herd replacement problem, compared to a general formulation of the multi-component replacement problem, is that the supply of replacements is not exogenous to the process but is actually generated by the process itself. Since many dairy farmers only use home-grown heifers as replacements (mainly because of the risk of introducing infectious diseases in the herd when heifers are bought at the market), the supply of heifers will be limited to those born in the herd. This further raises

the problem of deciding how many female calves to raise as potential replacements, which is also briefly discussed in this paper.

Most single-component studies in the literature dealing with dairy cows use dynamic programming and Markov decision processes in the determination of optimal replacement policies. Some of the models are very large. Thus, Arendonk and Dijkhuizen (1985) used a model with 174 000 states, reported by Arendonk (1988), and the model of Kristensen (1989) contained 180 000 states. Also, the multi-component problem can be formulated as a Markov decision process, but, since the states of all components should be considered simultaneously, the size of the total model will be far beyond computational capacity. Therefore, the need for approximate methods arises.

Ben-Ari and Gal (1986) discussed this problem and introduced a method called 'Parameter Iteration'. The idea is to approximate the total expected profit of the herd at a given composition by a function involving a set of parameters describing the relations between the total expected profit and the present herd composition. The parameters are determined in each situation by an iterative method.

Also, the method described in this paper is based on a parameter iteration technique, but the implementation of the idea is fundamentally different from that of Ben-Ari and Gal (1986). The main reasons for dealing with the problem again are as follows:

- (1) The approximation used by Ben-Ari and Gal (1986) is exact when the supply of heifers is unlimited (i.e. the opposite situation of the one studied).
- (2) The evaluation of the method seems insufficient in Ben-Ari and Gal (1986). No results have been presented showing the benefits of the method over the results from the underlying single-component model.
- (3) The single-component model of Ben-Ari and Gal (1986) was very simple, containing only 180 states.

In the present study, the problem is discussed under the assumption that no milk quota is present. An additional paper discussing the multi-component replacement problem under a milk quota will be considered later.

## 2 THE DAIRY HERD AS A MULTI-COMPONENT SYSTEM

In this section, the multi-component replacement problem in dairy herds is described and it is shown that, in principle, it may be formulated as an ordinary Markov decision process with known parameters.

A dairy herd with a limited maximum number of cows ( $L$ ) was considered.

Each cow and its successors are represented by a Markov decision process with known action and state spaces and known parameters (the *single-component model*). The reward of state  $i$  under the action  $a$  ( $a = \text{'keep'}$  or  $\text{'replace'}$ ) is denoted as  $r_i^a$  and the transition probability from state  $i$  at the present stage to state  $j$  at the following stage under action  $a$  is denoted as  $p_{ij}^a$ . The state  $i$  of a cow reflects the important characteristics of the cow (i.e. milk yield, age, reproductive status, etc.). Also, a state representing the absence of a cow (an empty stall) must be included.

The Markov decision process may be an ordinary process, as described by Howard (1960), or a Hierarchic Markov process, as described by Kristensen (1988). In both cases, there are iterative methods to determine an optimal policy under infinite horizon (i.e. an infinite number of stages) assuming an unlimited supply of heifers. Under the optimal policy the present value ( $f_i$ ) can be calculated, the total expected discounted rewards of the process starting in state  $i$  and running over an infinite number of stages following an optimal policy.

The single-component model of this study is identical to the model of Kristensen (1989). The state variables of that model are the genetic merit (5 classes), the lactation number (6 classes), the stage of lactation (18 classes), the milk yield of previous and present lactation (15 classes each), and the pregnancy status (8 classes). The model is of the hierarchic type with a total of 180 000 states.

A Markov decision process describing the entire multi-component system will now be defined. It will be referred to as the *multi-component model*. A state is defined from the values of  $L + H$  state variables describing cows and heifers in the herd. The  $L$  cow variables are defined by the states (in the single-component model) of the  $L$  cows (and empty stalls) of the herd.

Heifers are described by  $H$  state variables defined by age or pregnancy. The values of the heifer state variables are the numbers of heifers in each of the  $H$  states. State 1 represents heifers born at the previous stage, and state  $H$  represents down calving heifers.

Heifers in state  $1, \dots, n$  are young animals before heat detection is initiated, and the state number is equal to the age measured in stages. Heifers in state  $n + 1, \dots, H - 11$  are those being submitted for service. Also in this group the state number is equal to the age in stages. Conception is assumed to be independent over stages and animals. Thus, the transition probability  $p_c$  from any of the insemination states ( $n + 1, \dots, H - 11$ ) to state  $H - 10$  is assumed to be fixed. The states  $H - 10, \dots, H$  represent heifers in calf, and a heifer in state  $H - v$  ( $0 \leq v \leq 10$ ) is expected to calve  $v$  stages ahead.

As it appears, a cow state variable gives the state of one particular cow whereas a heifer state variable gives the number of heifers occupying one particular state. In the work of Ben-Ari and Gal (1986) the cow state variables were defined in the same way as the heifer variables in this study.

That was possible because the single-component model of their study was very small containing only 180 states. Thus, the number of cow state variables was also 180. In the present study the single-component model has got 180 000 states and followingly the number of variables would be the same if the formulation of Ben-Ari and Gal (1986) was used. In the present study, the number of state variables concerning cows is only  $L$  (i.e. the maximum number of cows).

An action in the multi-component model is a set of actions  $A = (a_1, \dots, a_L)$  defining the action for each individual cow. The admissible actions are restricted to those where the number of heifers used for replacement does not exceed the number available.

The reward  $R_I^A$  of state  $I$  under action  $A$  is given as

$$R_I^A = \sum_{n=1}^L r_{i_n}^a \quad (1)$$

where  $i_n$  is the (single-component) state of the  $n$ th component. In the model, the raising costs of heifers is not included. Instead, calves will be sold to heifers and replacements bought from heifers at market prices.

The transition probability is now considered from state  $I$  to state  $J$ . A state is defined by the cow state variables  $c_1, \dots, c_L$  and the heifer variables  $h_1, \dots, h_H$ . Transitions among the heifer states are independent of those among the cow states, so they are treated separately. The cow states are first considered, where the transition probability is just the product of all single-component transition probabilities involved:

$$P_{IJ}^A = p_{i_1 j_1}^A \times p_{i_2 j_2}^A \times \dots \times p_{i_L j_L}^A \quad (2)$$

The transition probabilities among the heifer states are determined solely by the combined heat detection and conception rate  $p_c$ . Thus, it can be seen that the overall transition probabilities exist and might be calculated.

All parameters of the multi-component model have now been defined and it is seen that it is just a usual Markov decision process with known parameters. Only computational capacity prevents the finding of an optimal replacement policy by the usual techniques.

### 3 THE IMPLEMENTATION OF THE PARAMETER ITERATION METHOD

#### 3.1 Theoretical considerations

In this section an approximative method to circumvent the capacity problems involved in the application of usual techniques is described.

If the actions of individual cows were independent of each others (i.e. at an unlimited supply of heifers) the total present value of the multi-component model  $F_I$  under an optimal policy would equal the sum of all individual present values  $f_i$ , i.e.

$$F_I = f_{i_1} + \dots + f_{i_L} \quad (3)$$

Since they are in fact not independent  $F_I$  is approximated by a function  $G$  involving a number of parameters  $g_1, \dots, g_n$  relating  $f_{i_1}, \dots, f_{i_L}$  and  $I$  to  $F_I$ . The question is now what kind of function should be preferred? Since the relation is linear in the independent case, Ben-Ari and Gal (1986) argued that it would be natural also in the dependent case to approximate by a linear function. However, since the linearity is caused by the independence assumption this does not seem to be a good choice. Instead, some logical characteristics that the function should possess will be argued for.

- (1) The total present value under a limited supply of heifers can never exceed the value under unlimited supply as expressed in eqn (3). Thus,  $F_I$  can be expressed as

$$F_I = f_{i_1} + \dots + f_{i_L} - G(I) \quad (4)$$

where  $G$  is a non-negative function.

- (2) The only way that  $F_I$  may be reduced compared to eqn (3) is by shortage of heifers. Thus, the reduction expressed by  $G$  in eqn (4) must be directly linked to the number of heifers in various states.
- (3) The reduction caused by shortage of heifers at a certain age (i.e. in a specific state) is decreasing with increasing number of heifers at that age. The reduction caused by a shortage at that age will vanish if the number available is sufficiently large. If no heifers are available at a certain age, the reduction from that age will decrease considerably if one is added. If a second one is added, the reduction will increase further, but not as much as for the first heifer.
- (4) The size of the reduction caused by a shortage of heifers depends on the composition of the cow herd. If many cows would be replaced in the case of unlimited supply, the reduction will be larger than if only few cows should be replaced. Thus,  $G$  must include some measure of total herd quality. Further, this relation is more prevalent in the case of a shortage of heifers near calving than in the case of a shortage of younger heifers, because the coherence between present and future quality is decreasing with increasing time interval. Thus, the reduction caused by a shortage of new-born heifers will be almost independent of the present cow herd quality, whereas the reduction caused by down calving heifers will be almost entirely linked to the cow herd quality.

- (5) Assume that  $n_1$  heifers of age  $a_1$  and  $n_2$  heifers of age  $a_2$  (where  $a_1 < a_2$ ) are available. The reduction caused by a shortage of heifers at any of the ages depends on the number of heifers as described in point (3), but if  $n_1 = n_2$ , and if the herd quality is assumed to be constant over time, the reduction caused by a shortage of heifers at age  $a_1$  only differs from the corresponding reduction caused by a shortage of heifers at age  $a_2$  due to the discount factor. Except for the discounting (the age  $a_2$  is closer to first calving than  $a_1$ ) the reductions are equal.

### 3.2 Choice of functions and parameters

One way to express the herd quality  $Q_t$  is to define it from the *future profitabilities* of the individual cows as defined by Kristensen (1987). It is the benefit (positive or negative) from keeping a cow for at least one additional stage compared to immediate replacement. In other words, a positive future profitability means that the optimal action in the single-component model is to keep, and a negative value means that the optimal action is to replace. The following formula is for (single-component) state  $i$ :

$$q_i^* = r_i^{a_1} + \beta \sum_j p_{ij}^{a_1} f_j - r_i^{a_2} - \beta \sum_j p_{ij}^{a_2} f_j \quad (5)$$

where the superscripts  $a_1$  and  $a_2$  refer to the actions keep and replace, respectively. The present value  $f_j$  of a cow in state  $j$  is known from the optimal solution to the single-component problem. It represents the total expected discounted rewards of a Markov decision process starting in state  $j$  and running over an infinite number of stages under an optimal policy. The interpretation of eqn (5) is that from the next stage an optimal policy will be followed, but at the present stage any of the actions may be chosen. The future profitability is then calculated as the difference in present value when the cow is kept for at least one stage compared to the present value of immediate replacement. The symbol  $\beta$  is the discount factor from a stage to the previous one.

A weakness of this definition of quality is that the estimated loss, if the future profitability is negative, is related only to a very short period (one stage). At the next stage it is assumed that a heifer is available, and if the future profitability is still negative, a replacement is assumed to take place. Thus, the future profitability of even the least efficient cow is numerically small (though negative). In other words, a negative future profitability does not indicate whether the cow is just in a temporary crisis or whether it is really not profitable in the long run either.

Instead of eqn (5), an alternative definition of quality shall be considered

$$q_i = \phi_{it} - r_i^{a_2} - \beta \sum_j p_{ij}^{a_2} f_j \quad (6)$$

where  $\phi_{it}$  is the present value (in the single-component model) of a cow which at stage  $t$  is in state  $i$  provided that it is kept at least until next calving. This present value is calculated recurrently according to

$$\phi_{it} = f_i$$

if a calving takes place in state  $i$ , and

$$\phi_{it} = r_i^{a_1} + \beta \sum_j p_{ij}^{a_1} \phi_{j,t+1}$$

if no calving takes place. The summation at the right-hand side of the equation is over all possible states at stage  $t + 1$ . The superscript  $a_1$  is the action 'keep' and  $a_2$  is 'replace'. Thus,  $q_i$  is the advantage (positive or negative) of keeping the cow at least until the next calving. In the calculation of  $\phi_{it}$  the absolute value of  $t$  is of no relevance. The calculation is just started in the states where a calving takes place, and the value for states representing other stages of lactation are then calculated backwards step by step beginning one stage before calving and ending one stage after the previous calving. The advantage of eqn (6) over eqn (5) is that the defined quality refers to a longer period (instead of just one stage). It therefore represents a more permanent characteristic of the animal avoiding a temporary crisis to result in a low-quality classification.

The herd quality in turn is defined as

$$Q_I = \sum_{n=1}^L q_{i_n} \quad (7)$$

In order to get an impression of the shape of the function  $G$ , the immediate loss from shortage of down calving heifers (i.e. heifers of state  $H$ ) will be considered. From the single-component model we know the future profitability (eqn (5)) of each individual cow in the herd at any time. If no heifers at all are available, the immediate loss will numerically equal the sum of all future profitabilities below zero (i.e. of all cows that would be replaced if the supply of heifers was unlimited). If one and only one heifer is available, the lowest ranking cow will be replaced if its future profitability is negative. The immediate loss thus numerically equals the sum of the remaining future

profitabilities below zero. Correspondingly, the immediate loss if 2, 3, 4 or more heifers are available may be calculated in a similar way. If the number of heifers available exceeds the number of cows having negative future profitabilities the loss will be zero.

By simulation, it is possible to generate a large number of herd combinations and thus a large number of joint observations of total herd qualities and immediate losses if 0, 1, 2, 3, ... heifers of state  $H$  are available. By analysis of such a simulated material it was found that a good fit was obtained by the function

$$g_0(h_H, Q_I) = a \exp(bh_H + cQ_I) \quad (8)$$

where  $g_0(h_H, Q_I)$  is the expected immediate loss if the number of heifers just about to calve is  $h_H$ , and the total herd quality is  $Q_I$ . The symbols  $a$ ,  $b$  and  $c$  are parameters to be estimated. It is expected that  $a$  will be positive, and  $b$  and  $c$  will be negative. It appears that eqn (8) becomes linear in  $h_H$  and  $Q_I$  by using logarithms. Therefore, the values of  $a$ ,  $b$  and  $c$  may be determined by ordinary least-squares regression.

From these results the expected loss from shortage of heifers in state  $H - 1$  (i.e. heifers expected to calve one stage ahead) are now considered. From eqn (8) the expected discounted loss is

$$g_1(h_{H-1}, Q_I) = \beta E(g_0(h_{H-1}, Q_J) | I) \quad (9)$$

where the stochastic variable  $Q_J$  is the herd quality at the following stage given a present herd quality of  $Q_I$ . The number of heifers in state  $H$  at the following stage equals the number in state  $H - 1$  at the present stage. If eqn (8) is substituted into eqn (9), then

$$g_1(h_{H-1}, Q_I) = \beta E(a \exp(bh_{H-1} + cQ_J) | I) \quad (10)$$

In a similar way, the expected discounted loss from shortage of heifers in any other state of pregnancy  $H - 10, \dots, H - 2$  may be calculated. The central elements are a discount factor corresponding to the time gap until the heifers of a state are expected to calve and the expected loss at that time given the present herd quality.

The expected loss from heifers not yet pregnant is more complicated to calculate, because the expected number of these heifers to calve  $v$  stages ahead is not equal to the number of heifers in a specific state. Young heifers of the same age will typically not conceive at the same time, and heifers of different ages may conceive at the same time. Recalling that states  $1, \dots, n$  represent young heifers before heat detection is initiated, and states  $n + 1, \dots, H - 11$  represent heifers under insemination, it can be concluded that the total number,  $H_{11}$ , of heifers from the insemination states to calve 11 stages ahead is binomially distributed with the parameters

$N = h_{n+1} + \dots + h_{H-11}$  and  $p = p_c$ . Accordingly, the expected value and variance of the total number of heifers to calve 11 stages ahead are

$$E_{11}(H_{11} | I) = p_c \sum_{i=n+1}^{H-11} h_i \quad (11)$$

$$V_{11}(H_{11} | I) = p_c(1 - p_c) \sum_{i=n+1}^{H-11} h_i$$

If the total number of heifers to calve 12 stages ahead,  $H_{12}$ , is looked at it is found that the number of heifers from state  $n$  is binomially distributed with the parameters  $h_n$  and  $p_c$ . The number of heifers to calve 12 stages ahead from the present insemination states is also binomially distributed with the parameters  $h_{n+1} + \dots + h_{H-12}$  (heifers from state  $H-11$  are not included because if they do not conceive at the present stage—calving 11 stages ahead—they are culled) and  $(1 - p_c)p_c$  (i.e. the probability that they conceive at the next stage provided that they do not conceive at the present stage). Thus, the expected value and variance of the total number of heifers to calve 12 stages ahead are

$$E_{12}(H_{12} | I) = h_n p_c + (1 - p_c) p_c \sum_{i=n+1}^{H-12} h_i \quad (12)$$

$$V_{12}(H_{12} | I) = h_n p_c (1 - p_c) + (1 - p_c) p_c (1 - (1 - p_c) p_c) \sum_{i=n+1}^{H-12} h_i$$

Continuing in the same way it is found that the total number of heifers to calve  $v$  stages ahead ( $v > 10$ ) is a sum of binomially distributed random variables having the expected value and variance as follows:

$$E_v(H_v | I) = \sum_{i=n-v+12}^{n'} p_c(1 - p_c)^{(i-n+v-12)} h_i + p_c(1 - p_c)^{v-11} \sum_{i=n+1}^{H-v} h_i \quad (13)$$

$$V_v(H_v | I) = \sum_{i=n-v+12}^{n'} p_c(1 - p_c)^{(i-n+v-12)} (1 - p_c(1 - p_c)^{(i-n+v-12)}) h_i$$

$$+ p_c(1 - p_c)^{v-11} (1 - p_c(1 - p_c)^{v-11}) \sum_{i=n+1}^{H-v} h_i$$

where  $n' = \min \{n, H - v\}$ . The limits of the first sum in the equation are from the youngest possible heifers to calve  $v$  stages ahead to *either* the oldest heifers presently not being observed for heat *or* the oldest possible heifers to calve  $v$  stages ahead (whichever is lowest). The limits of the second sum are from the youngest heifers presently being observed for heat to the oldest heifers in the insemination states that are not being discarded if they do not become pregnant to calve  $v$  stages ahead (or before). If for a value of  $v$ , the lower limit of a sum is higher than the upper limit, the sum will vanish.

The total reduction  $G(I)$  in present value caused by shortage of heifers is calculated as the sum of expected losses from shortage of heifers expected to calve different stages ahead, i.e.

$$G(I) = \sum_{v=0}^{N'} \beta^v E_v(a \exp(bH_v + cQ_J) | I) \quad (14)$$

where the random variables  $H_v$  and  $Q_J$  are the total number of heifers calving and the herd quality, respectively,  $v$  stages ahead given the present (multi-component) state  $I$ . The correct value of  $N'$  is infinity, but for practical purposes it is reasonable to let  $N'$  be the maximum age (in stages) that a new-born heifer may possibly calve under the defined insemination and culling policy.

In order to calculate  $G(I)$ , how to evaluate the expected value at the right-hand side of eqn (14) must be considered. In other words, the distribution of the time series of observed herd qualities at successive stages must be known. It is obvious that if the herd quality is low at the present stage, it must be expected to be low at the following stage too. Further, a large number of heifers to calve at the present stage is assumed to imply a higher herd quality at the next stage, because of the possibilities of replacement. A simple way to model this property is to define the time series as follows, where  $Q_t$  is the herd quality at stage  $t$ :

$$\begin{aligned} Q_{t+1} &= m + fh_H + e_{t+1} \\ \text{and} \quad e_{t+1} &= de_t + \varepsilon_{t+1} \end{aligned} \quad (15)$$

where  $m$  is the average value under the (multi-component) policy followed if no heifers were available,  $f$  is a parameter describing the marginal improvement caused by an additional heifer,  $d$  is an autoregression coefficient and the residuals  $\varepsilon_t$  are assumed to be mutually independent and normally distributed with zero mean and a standard deviation of  $\sigma$ . The values of  $m, f, d$  and  $\sigma$  will depend on the (multi-component) policy. Thus, at an intensive culling,  $m$  is assumed to be higher (i.e. better herd quality). The value of  $d$  is assumed to be lower because many replacements will decrease

the correlation over stages. Finally, an intensive culling is expected to decrease the random variation  $\sigma$ .

The expected value and the variance of the herd quality  $v$  stages ahead, given the present state  $I$ , are calculated as follows:

$$\begin{aligned} E_v(Q_J | I) &= m + fE_v(H_v | I) + d^v(Q_I - m - fh_H) \\ V_v(Q_J | I) &= f^2 V_v(H_v | I) + \sigma^2 \frac{1 - d^{2v}}{1 - d^2} \end{aligned} \quad (16)$$

where the conditional expectation and variance at the right hand sides are known from eqn (13).

Given the model in eqn (11), an approximate value of the right-hand side of eqn (9) can be calculated. For a given (multi-component) state  $I$ , the expression  $bH_1 + cQ_J$  is normally distributed with an expected value of  $bh_{H-1} + c(m + fh_{H-1} + d(Q_I - m - fh_H))$  and a variance of  $c^2\sigma^2$ . Consequently, the distribution of  $\exp(bH_1 + cQ_J)$  is log-normal with an expected value of  $\exp(bh_{H-1} + c(m + fh_{H-1} + d(Q_I - m - fh_H)) + c^2\sigma^2/2)$ . For all values of  $v < 11$  the expression  $bH_v + cQ_J$  is normally distributed under the assumptions used) and therefore the distribution of the exponential value is log-normal) with the expectation and variance given as

$$\begin{aligned} E_v(bH_v + cQ_J | I) &= bh_{H-v} + c(m + fh_{H-v} + d^v(Q_I - m - fh_H)) \\ V_v(bH_v + cQ_J | I) &= c^2\sigma^2 \frac{1 - d^{2v}}{1 - d^2} \end{aligned} \quad (17)$$

For values of  $v \geq 11$  the situation is more complicated since  $H_v$  in this situation is a random variable, which is a sum of several binomially distributed random variables. Therefore, the expression  $bH_v + cQ_J$  is not normally distributed. Since, however, the normal distribution is usually a good approximation of a binomial distribution, and further  $H_v$  is a sum of several random variables, the expression is assumed to be approximately normally distributed. The mean and variance may be calculated as

$$\begin{aligned} E_v(bH_v + cQ_J | I) &= bE_v(H_v | I) + cE_v(Q_J | I) \\ V_v(bH_v + cQ_J | I) &= b^2 V_v(H_v | I) + c^2 V_v(Q_J | I) + 2bcfV_v(H_v | I) \end{aligned} \quad (18)$$

where the conditional means and variances of the right-hand side are known from eqns (13) and (16). Accordingly

$$E_v(\exp(bH_v + cQ_J | I)) = \exp(E_v(bH_v + cQ_J | I) + \frac{1}{2}V_v(bH_v + cQ_J | I))$$

If this equation is used in eqn (14) the following is obtained:

$$G(I) = a \sum_{v=0}^{N'} \beta^v \exp(E_v(bH_v + cQ_J | I) + \frac{1}{2}V_v(bH_v + cQ_J | I)) \quad (19)$$

The (multi-component) future profitability of a cow is

$$\pi_i = q_i^* + G(I^1) - G(I^2) \quad (20)$$

where  $I^1$  and  $I^2$  are the multi-component states if the cow is kept or replaced, respectively, and  $q_i^*$  is the future profitability defined in eqn (5) for the single-component model. An approximately optimal policy of the multi-component model is given by the optimal policy of the single-component model combined with the parameters of the function  $G$ . A cow is replaced if the future profitability of eqn (14) is negative and kept otherwise. Each time the lowest ranking cow has been replaced, the future profitabilities of the remaining cows must be recalculated under the new herd quality and the reduced number of heifers caused by the replacement.

### 3.3 An approximate solution

Having chosen the functions and parameters, the steps involved in the determination of an approximately optimal policy of the multi-component model may be described:

- (1) Calculate an optimal policy of the single-component model by usual methods.
- (2) Estimate the parameters  $a, b$  and  $c$  of eqn (7) from a simulated data set using an arbitrary policy. The parameters  $m, d, f$  and  $\sigma$  of eqn (11) are also estimated.
- (3) Simulate a time period using the present parameters  $m, d, f$  and  $\sigma$  to define the policy to be followed. Calculate the economic result of the simulated period and estimate new values of  $m, d, f$  and  $\sigma$  from the simulated data set.
- (4) Repeat step (3) until the results have stabilised.

### 3.4 Culling of heifers

In the previous sections, the problem of optimal replacement of cows, when a given number of heifers in different states are available, has been considered. The opposite problem of whether a heifer in a specific state should be sold or raised for future milk production in the herd is now considered. In that connection, the *production value*  $v(i, I)$  of a heifer in state  $i$  given the multi-component state  $I$  is defined as

$$v(i, I) = G(I_1) - G(I) \quad (21)$$

where  $I_1$  is the multi-component state if one heifer in state  $i$  is sold. The value of  $G(I)$  is calculated from eqn (19). The production value expresses the expected future contribution of the heifer to net returns from the production of cows. This value is calculated from the point of view of the *milk producer*.

Further on, the *alternative value* is defined as the present market value of the heifer plus the expected discounted costs of raising the heifer from present age to first calving minus the discounted price of a down calving heifer. The alternative value expresses the gain (positive or negative) from selling the heifer immediately compared to keeping it until just before calving and then sell it or include it in the cow herd (which ever is best at that time). The alternative value is calculated from the point of view of the *heifer producer*.

However, the whole idea of the multi-component model is that the heifer producer and the milk producer is one and the same. On considering the sale of a heifer, the production value should be compared to the alternative value. If the alternative value is higher it is profitable to sell the heifer; otherwise, it should be kept for future replacement.

## 4 TEST OF THE METHOD

### 4.1 Material and methods

In order to test the method described in the previous section a single-component dairy cow replacement model and a stochastic simulation model of a dairy herd is needed. As mentioned in section 2, the replacement model used is the one described by Kristensen (1989). The simulation model is the one used by Kristensen and Thyssen (1991). It includes, for each cow, the same traits as the replacement model. Further, it includes the heifers of the herd. The main characteristics of the simulation model are shown in Table 1.

A standard set of prices and herd level of milk yield was defined as in Table 2. In order to test the method under various conditions, 100 sets of alternatives have been generated by a random number generator. In each set of conditions the individual prices and level of milk yield were drawn independently from uniform distributions over intervals defined from the original values of Table 2  $\pm 15\%$ .

Under each of the 100 sets of conditions the method of section 3.3 was applied. In the simulation of step (2), the cows were ranked according to their single-component future profitability as defined in eqn (5). Thus, a cow is replaced if the future profitability is negative *and* a heifer is available. In the simulation of step (3), the cows were ranked according to their multi-component future profitability as defined in eqn (20). (The ranking was recalculated each time a replacement was performed). In all cases a herd of 100 cows (as a maximum) is simulated over a period of 100 years in order to decrease the random variation on results.

Step (1) of the method gives the economic result if an optimal policy is

**TABLE 1**  
Main Characteristics of the Simulation Model<sup>a</sup>

<i>Cows</i>	
Number of 4-week stages per lactation	11–18
Maximum number of lactations per cow	6
Forced replacement if not pregnant before	238 days
Stochastic state variables	
Breeding value (milk yield) of father	5 classes
Milk yield, previous lactation	15 classes
Milk yield, present lactation	15 classes
Length of calving interval	8 classes
Deterministic state variables	
Lactation stage	18 classes
Lactation number	6 classes
Total number of states (approximately)	180 000
Probability of a calving to result in a surviving heifer	0.45
<i>Heifers</i>	
State variable	
Age or reproductive status	41 classes
Minimum age of first breeding	56 weeks
Probability of conception per stage	0.33
Age of disposal of open heifers	116 weeks

<sup>a</sup> From Kristensen and Thysen (1991).

**TABLE 2**  
Standard Prices and Herd Level of Milk Yield

Prices (Dkr)	
Milk (kg FCM <sup>a</sup> )	2.40
Basic feed (SFU <sup>b</sup> )	1.30
Feed for milk production (SFU)	1.45
Calf	1400.00
Heifer	9000.00
Young cow (kg live weight)	11.50
Older cow (kg live weight)	11.00
Interest rate (corrected for tax and inflation, %)	3.00
Herd level of milk yield (week 1–40, 1st lactation)	5800.00

<sup>a</sup> Fat corrected milk.

<sup>b</sup> Scandinavian feed unit.

followed and an unlimited supply of heifers is available. This result thus gives an upper bound of the economic result using the new method. A lower bound is given by the economic result of the simulation of step (2), where the single-component future profitability is used as replacement criterion under limited supply of heifers. The new method should do better than that in order to be relevant.

The standard conditions are used for calculating the production values of heifers in herds of varying composition.

## 4.2 Results

In all 100 sets of conditions step (3) of the method was run three times in order to see when the results stabilised. The economic results, however, did not improve by running step (3) more than once. On the other hand, the values of the parameters  $m$ ,  $d$ ,  $f$  and  $\sigma$  often (but not always) changed from step (2) to step (3), but only slightly from first to second run of step (3). All results in the following are taken from the second run of step (3). In Table 3 the parameter estimates under standard conditions are shown.

Denote as  $O_i$  ( $i = 1, \dots, 100$ ) the economic result in Danish kroner (Dkr) per cow per year (revenues from milk, calves and culled cows minus the costs of feeds and heifers) under an optimal policy for the  $i$ th set of conditions assuming unlimited supply of heifers. These results are the expected values calculated directly from the functional equations of the single-component Markov decision process. The simulation results for the  $i$ th set of conditions under limited supply of heifers are denoted as  $S_i$  and  $M_i$  using the single- and multi-component future profitabilities, respectively. Unlike  $O_i$ ,  $S_i$  and  $M_i$  are

**TABLE 3**  
Parameter Estimates of the Models in Eqns (8) and (15) under the Standard Conditions of Table 2

Parameter	Symbol	Step 2	Step 3	
			First run	Second run
Level	$a$	$3.12 \times 10^7$		
Effect of heifers	$b$	$-0.544$		
Effect of herd quality	$c$	$-0.670 \times 10^{-4}$		
$R^2$ of eqn (8)		0.71		
Basic level	$m$	$1.43 \times 10^5$	$1.45 \times 10^7$	$1.50 \times 10^7$
Autoregression coefficient	$d$	0.762	0.784	0.791
Effect of heifers	$f$	1675	1946	1824
Standard deviation	$\sigma$	$1.17 \times 10^4$	$1.18 \times 10^4$	$1.14 \times 10^4$
$R^2$ of eqn (15)		0.64	0.68	0.68

TABLE 4

Simulation Results (Dkr per Cow per Year Calculated as Revenues from Milk, Calves and Culled Cows Minus Costs of Feeds and Heifers) using Single and Multi-Component Models Compared to Results from Optimal Policies at Unrestricted Heifer Supply

Method	Symbol <sup>b</sup>	Percentile <sup>a</sup>					Mean
		1	5	50	95	100	
Optimal policy	$O_i$	5128	6553	9402	13 014	14 665	9540
Single-component model <sup>c</sup>	$S_i - O_i$	-1482	-867	-109	-29	7	-200
Multi-component model <sup>c</sup>	$M_i - O_i$	-1059	-589	-87	-9	24	-140
Benefit of multi-component model	$M_i - S_i$	-59	-48	34	218	423	60

<sup>a</sup> Calculated for each variable independently of the others.

<sup>b</sup> Defined in the text.

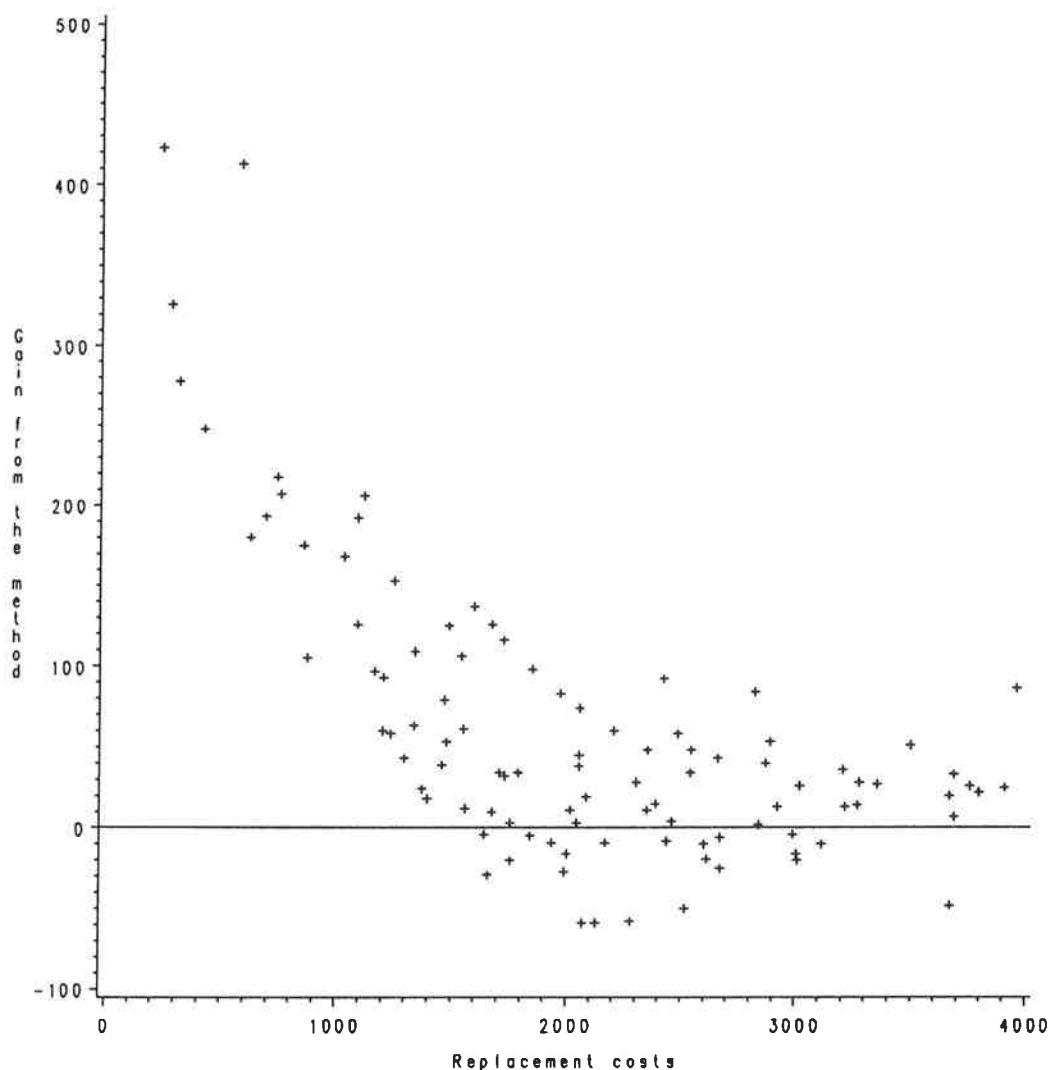
<sup>c</sup> Deviation from unrestricted optimal policy.

not expected values, but only estimates of the true expectations. Therefore, a certain (limited) variation around the true values is unavoidable. In Table 4 the percentiles, extremes and means over the 100 sets of conditions are shown for  $O_1, \dots, O_{100}$  as well as for the differences  $S_1 - O_1, \dots, S_{100} - O_{100}$  and  $M_1 - O_1, \dots, M_{100} - O_{100}$ . Finally, the same statistics are shown for the differences  $M_1 - S_1, \dots, M_{100} - S_{100}$  representing the benefits of the multi-component model over the single-component model.

As it appears, the results from the multi-component model are on average 60 Dkr better per cow per year than when the single-component future profitabilities are used as replacement criterion. The results using the single-component criterion are on average 200 Dkr below the unrestricted optimal solution and those using the multi-component criterion are on average 140 Dkr below the unrestricted optimal solution per cow per year.

As it appears from Table 4, the benefit of the multi-component method over the single-component varies considerably over the 100 sets of conditions. It should be expected that the benefit is highest in situations where the supply of heifers is smaller than the optimal need for replacements. The most important factor determining the level of replacement is the price difference between a heifer and the carcass value of a cow (Kristensen & Østergaard, 1982). Almost equivalent to this difference is the *replacement costs* defined as the price of a heifer minus the slaughter value of a young cow and the value of a calf.

In Fig. 1, the benefit of the method is plotted against the replacement costs under each of the 100 sets of conditions. As expected, the benefit is very high in situations with low replacement costs, where the optimal level of replacement is high, and the heifers available do not suffice. In situations with low replacement costs the benefit seems almost to vanish. In some cases



**Fig. 1.** Plot of the benefit (Dkr) of the multi-component model over the single-component model against the replacement costs (Dkr—see the text for definition).

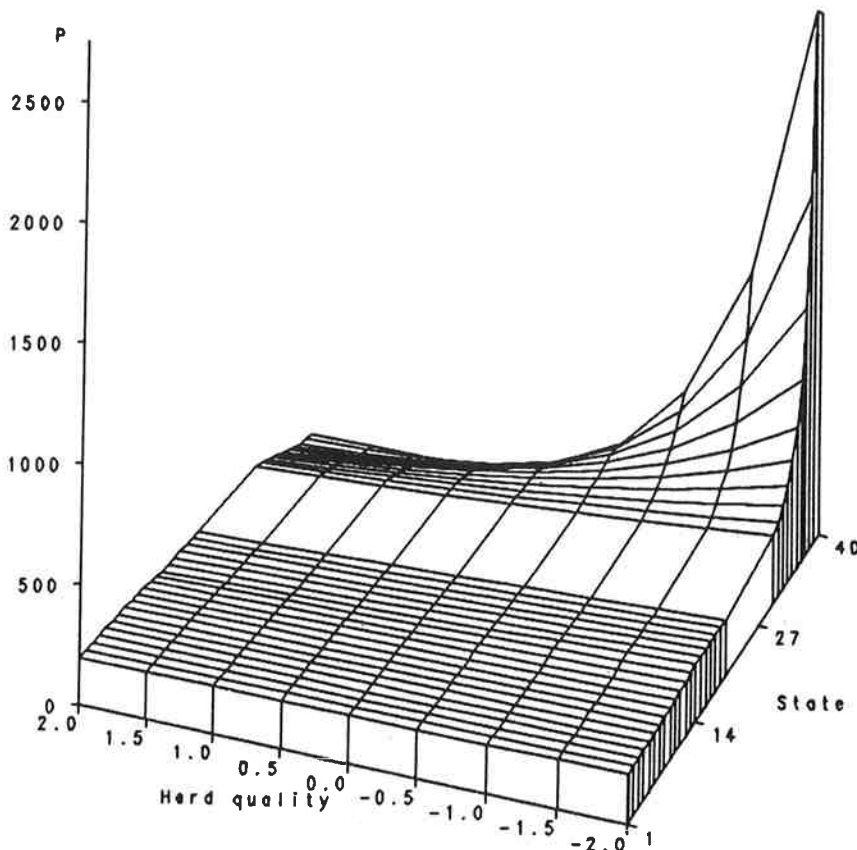
the benefit is even negative, but, since the values represent the results of stochastic simulation, at least some negative values should be expected when the true value is numerically small though non-negative. By repetitive simulation runs under fixed conditions the standard deviation of a simulation result was determined to be around 30 Dkr. Since the benefits in Fig. 1 represent differences between two results the standard deviation in the direction of the y-axis becomes 42 Dkr. The negative values are therefore easily explained as results of random variation around a non-negative true value.

In order to study the production values of heifers in relation to present herd quality and heifer state (age or pregnancy status) a heifer stock was designed in a manner that rather precisely results in the same expected

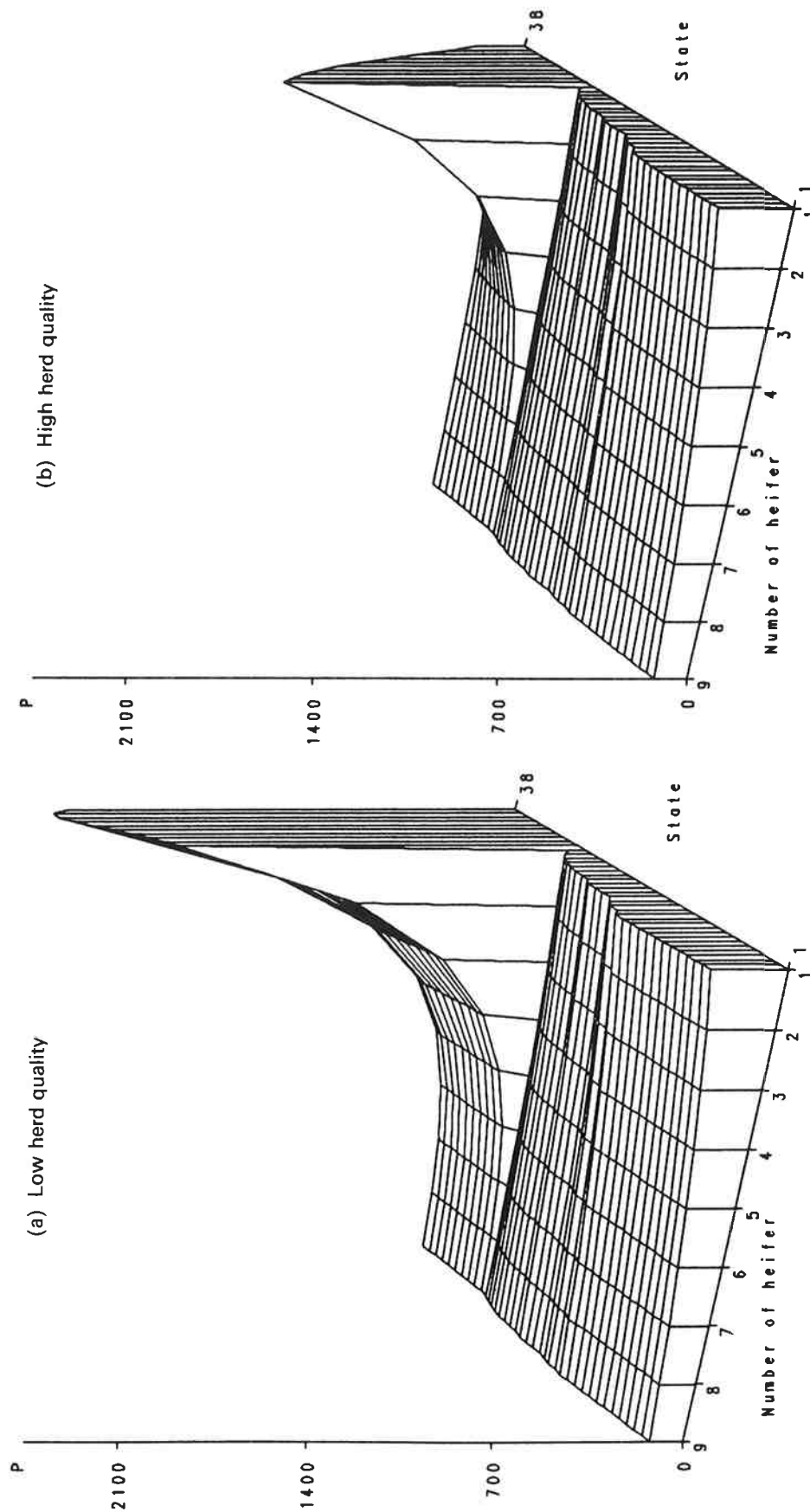
number of heifers to calve at each future stage 40 stages ahead. Under the standard conditions the herd quality was successively varied from the average value minus 2 units of standard deviation in steps of 0.5 units to the average value plus 2 units. For each heifer state the production value of the last heifer was determined as the loss in expected discounted future net returns from cows if exactly one heifer from the state was culled. The parameter estimates of Table 3 (step (3), first run) were applied.

In Fig. 2 the production value is plotted against heifer state and herd quality. As expected the production value depends heavily on the herd quality for heifers which are soon going to calve. For young heifers the effect vanishes.

In Fig. 3 the production value of the last heifer in a state is plotted against the number of heifers in the state and the state number. The basis is the same heifer composition as used in Fig. 2, but the number of heifers has been varied from 1 to 9 one state at a time. The calculations have been performed for 2 levels of present herd quality resulting in two different plots.



**Fig. 2.** The production value (P) of a heifer in Dkr as a function of present herd quality (unit: standard deviation, 0 = mean value) and heifer state (age or pregnancy status). State 1 represents newly born heifers and state 40 represents heifers just about to calve. The basic heifer stock is constructed so that the supply of heifers per stage is constant.



**Fig. 3.** Plot of production value ( $P$ ) in Dkr of the last heifer in a state as a function of the number of heifers in the state and the state number. States 1–13 represent young heifers before heat detection is initiated, states 14–28 represent heifers under insemination and states 29–38 represent pregnant heifers.

In both cases the plot is divided into three sections, which are states 1–13 representing young heifers not yet observed for heat, states 14–28 representing heifers under insemination, and states 30–39 representing heifers in calf. In the first two sections the effect of the number of heifers in a state is much smaller than for pregnant heifers. The reason is that for these young animals, the heifers of a particular state are not expected to calve at the same time because of the random variation in heat detection and conceiving. Thus, few heifers in one state is to a large extent compensated by sufficient heifers in other states. A similar compensation is not possible for heifers in calf and therefore shortage of heifers in a state is far more critical in those cases.

The reason for the lower production value of heifers under insemination compared to young heifers before insemination is that, for instance, five heifers in an insemination state are relatively more than the same number in one of the states from 1 to 13. The expected number of heifers in the youngest states is the average number of heifers born at a particular stage (4-week period). The expected number of heifers in an insemination stage is lower because some of the heifers at the age in question have already conceived and thus are transferred to a pregnancy state.

At high herd quality, the future quality is expected to be lower than it is at present, and therefore the production value of young heifers in calf is higher than for down calving heifers. At low quality the situation is opposite as it appears from Fig. 3.

## 5 DISCUSSION

This study is a contribution to the practical solution of the multi-component replacement problem in dairy cattle, where the limited supply of heifers as mentioned by Ben-Ari *et al.* (1983) complicates the problem compared, for example, to replacement of industrial items. An approximate method is suggested, since the calculation of an exact solution is prohibitive. Even the approximate method is very time consuming on the computer. The main reasons are the size of the single-component model used and the simulations of steps (2) and (3). If a single-component model of that size is used, we must conclude that *at present* the calculations are too comprehensive for direct practical application on a dairy herd. On the other hand, the calculations were performed on a (powerful) PC, which in a few years probably is a standard equipment on a commercial dairy farm. When further multi-tasking computer systems come into general use, the time spent on a single job is not so important because the computer may be used simultaneously for other purposes. Therefore, a method as the one described may very well be applied in a *future* decision support system concerning replacement in dairy herds. In

this connection it should be noticed, that as long as the price conditions are the same, the optimal policy is represented solely by the optimal solution to the unrestricted single-component problem combined with the parameter estimates concerning  $a$ ,  $b$ ,  $c$ ,  $m$ ,  $d$ ,  $f$  and  $\sigma$  of the function  $G(I)$  in eqn (20). Therefore, a new optimal policy of the multi-component problem only has to be calculated if prices change.

It is not possible to compare the results to those derived from exact solutions to the multi-component model. However, the alternative to the multi-component model is to use the future profitabilities from the single-component model as replacement and ranking criterion, and we are able to compare the results to this alternative. From Fig. 1 and Table 4 it can be concluded, that for low replacement costs (i.e. when the need for heifers exceeds the supply) the multi-component model improves the economic result considerably compared to the usual single-component model. For high replacement costs (where the supply of heifers is sufficient) the result is almost the same no matter if the single or multi-component model is applied. The reason is that for increasing number of heifers the multi-component future profitability of eqn (20) converges towards the value of the single-component future profitability of eqn (5) thus making the two criteria equivalent if the number of heifers is sufficiently large.

In such a situation the only advantage of the multi-component model is the information concerning culling of heifers as described in section 3.4. The information is given as the production value of the heifer, and the major force of the method is that it places the heifer in the herd environment where it belongs. In a herd where heifers are bought at the market, the value of a heifer is just the market price which only depends on the state of the animal in question. On the other hand, if only home-grown heifers are used as replacements (for the reasons mentioned in section 1) the value of a heifer can not be determined by calculations only relating to that particular animal. Instead, the following questions will have to be considered: How many other heifers at similar age have we got? What is the future need for replacements (expressed by the current herd quality)? The value of the heifer heavily depends on the answers to these questions, and this dependency is directly taken into account in the multi-component model.

An examination of the effects in Figs 2 and 3 confirms that the production value of pregnant heifers depends very much on the present herd quality and the number of heifers in a state. For younger heifers the effect of present herd quality almost vanishes because of the decreasing correlation between present and future quality over increasing time lag. The effect on number of heifers in the particular state in question is still present but to a much smaller extent than for pregnant heifers. The production value of the heifers may be calculated in any situation, and in combination with the alternative value as

defined in section 3.4, it is very important in the decisions concerning how many heifers to raise for future replacement. Also, in a situation with limited housing capacity the information is relevant. Since the value of this information concerning heifers is even more important in a situation with high replacement costs (because the supply of heifers in those cases exceeds the demand), it is relevant to use the multi-component model even in a situation with high replacement costs, where the benefit in the cow herd is small.

As concerns the goodness of fit of the approximations in eqns (8) and (15), the regression analyses resulted in  $R^2$  values of 0.71 in eqn (8) and in eqn (15) the values varied from 0.64 to 0.68 under the standard conditions (Table 3). The estimates of the autoregression coefficient of eqn (15) varied from 0.76 to 0.79 (standard conditions, Table 3) showing a high degree of autocorrelation in herd quality over time. All effects in the models were highly significant, and the parameters were very precisely estimated. Thus, the overall impression is that the models used in the approximations seem to fit quite well.

The supply of heifers has been identified as a limiting restraint on the replacement problem in many dairy herds. The basic idea of the multi-component approach is to consider in what way the limiting restraint logically affects the known optimal solution to the unrestricted problem. Then the influence of the restraint is approximated by a function  $G(I)$  having the desired logical properties, and finally the parameters of the function are estimated from simulated data.

There are several other limiting restraints on the replacement problem in dairy herds. The most obvious one at the time being is the milk quota, but also feed supply and/or labour might be considered. Similar problems exist in other multi-component systems as for instance herds of other animal species. A much wider range of problems involving replacement combined with general resource allocation is then faced, and it is relevant to consider whether the multi-component approach of this paper also might be used in a solution of such problems. It seems natural to expect that the *basic* idea of this study is applicable in any multi-component replacement problem subjected to some limiting restraint, but the actual choice of the function  $G(I)$  depends on the specific problem. Thus, the kind of function used in this study may not apply to other problems.

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# Applicational perspectives

## 1. Introduction

The applicational perspectives of the techniques discussed in this thesis cover at least the following three areas:

- 1) *Research*. The models may be used as tools in the identification of traits which must be considered in the replacement decision. Furthermore the influence of prices and other conditions on the optimal policies may be studied.
- 2) *Development of methods to be used in practice*. At present the Markov decision programming techniques are too time- and memory-consuming to be used in practice in connection with a commercial animal herd. However, the models may be used in the development of more simple operational methods.
- 3) *Direct application in commercial herds*. Within the foreseeable future direct application of the techniques may become realistic.

## 2. Application in research

Most of the studies mentioned in the previous chapters have only been used for research purpose. In dairy cows results have been obtained concerning the influence of prices and herd level of milk yield on the optimal policies (Kristensen and Østergaard, 1982; van Arendonk, 1985). The influence of changes in reproduction have been studied by van Arendonk and Dijkhuizen (1985), and the influence of seasonal variation in prices and performance has been studied by van Arendonk (1986). Kristensen (1987) studied the influence of the genetic class (defined from the breeding value of the father) on the length of the herd life time of a cow. The effect of a milk quota was studied by Kristensen (1989). The effect of clinical mastitis has been studied by Stott and Kennedy (1990) and at present by Houben et al. (1992).

In sows, similar studies were carried out by Huirne et al. (1988) in order to determine the influence of prices, herd level of litter size and time interval from weaning to conception.

Another kind of studies have had the objective of studying the economic value of culling information. An example in dairy cows is Kristensen and Thysen (1991a) (Chapter X), who studied the problem in the presence and absence of a milk quota. Other examples in dairy cows are Dijkhuizen and Stelwagen (1988) and Marsh et al. (1987). In sows a study was carried out by Dijkhuizen et al. (1989).

## 3. Development of methods to be used in practice

As long as direct application of the techniques in commercial herds is prohibitive, they may be used indirectly. One possibility is to determine an optimal policy under a set of standard conditions and apply that policy in other herds ignoring the individual deviations from the standard conditions. Thus results from Kristensen and Østergaard (1982) show that the *ranking* of animals is very stable towards changes in prices. The consequences of using a ranking determined under standard conditions under other conditions may be determined by comparing the results under the standard ranking to results under optimal ranking under the conditions of the individual herd. The work of Kristensen and Thysen (1991b) (Chapter XI) studies this problem in the presence and absence of a milk quota. Also van Arendonk (1988) has suggested this method.

The optimal ranking determined by the Markov decision programming techniques may also be compared to other more operational ranking criteria as it was done by Kristensen and Thysen (1991b).

No similar studies known to the author have been carried out in sows.

#### 4. Direct application in commercial herds

Even though the techniques presented in this thesis are *at present* prohibitive for direct application the situation may very well change in the future. Powerful personal computers may very soon become standard equipment of commercial herds. Furthermore, the operating systems develop, and multi-tasking systems have already been introduced in the personal computer environment. In such systems the time spent on a single job is not so crucial, because the computer may simultaneously be used for other purposes. In Figure 1, relative performances of personal computers used in the research behind this thesis have been compared. The development over these few years clearly illustrates that what is prohibitive today may very well be possible tomorrow.

If the relative performance of personal computers will continue to improve over the following years, the time is not far ahead when a direct application of the techniques of this thesis is *technically* possible. However, this does not necessarily imply that it is also appropriate. It is in no way impossible that the applicational scope of the techniques also in the future will be limited to the areas mentioned in Sections 2 and 3 of this chapter.

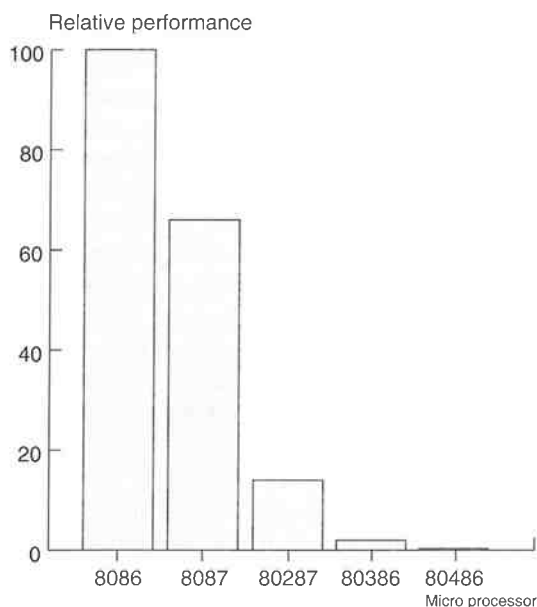


Figure 1. The approximate relative performance of selected personal computers in the determination of an optimal replacement policy for the model described by Kristensen (1989). The computer with a 8086 micro processor was purchased in 1984, and the 80486 computer was purchased in 1991.

#### 5. Conclusion

We may conclude that *at present* the applicational scope of the techniques is limited to studies of the traits and conditions that influence the optimal replacement policies and to comparative studies in the development of operational methods to be used in commercial herds. *In the future* it will probably be technically possible to use the techniques directly in commercial herds. Whether it is appropriate or not will depend on the results of the comparative studies, and no final conclusion can be drawn from the results of this thesis.

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X



## Economic Value of Culling Information in the Presence and Absence of a Milk Quota

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In order to determine the value of culling information, the net returns to housing, labour and management were calculated analytically using three different replacement policies in the presence and absence of a milk quota. The conclusions were that in the absence of a milk quota there are considerable benefits from using a decision support system, but under a quota the benefits are negligible if compared to the very simple policy of only replacing cows which fail to conceive within 238 days. If a system based on calculations assuming no quota is used under a quota the dairy farmer will be directly misinformed. Decision support systems for culling should be specifically designed for the quota situation, where reductions of costs are the most important means for improving herd net returns. Emphasis should therefore be put on information that support reductions on average costs of keeping a cow. *Key words: Dairy cow, replacement, decision support.*

### INTRODUCTION

The dairy cow replacement problem has been the object of several studies in the literature. The preferred tool for optimization has been stochastic dynamic programming, and very detailed models have been developed by van Arendonk (1985, 1986), van Arendonk & Dijkhuizen (1985) and Kristensen (1987, 1989). The works mentioned have resulted in profound knowledge of the nature of the problem as well as the factors influencing the optimal replacement policy. Further Kristensen (1989) has discussed the problem of adjusting such models to fit a situation with a milk quota.

Thus from a theoretical point of view the replacement problem should be regarded as solved to a satisfactory degree. The detailed models, however, are at present not suitable for practical use in commercial dairy herds because of the extreme amount of computing time required to get an optimal policy. Therefore alternative approaches must be considered to provide the dairy farmer with culling information. In a paper of van Arendonk (1988) one method has been described, but the situation with a milk quota has not been considered, and the benefits from the method has not been compared to those of other alternatives.

In Denmark a decision support system for culling of dairy cows in commercial dairy herds is under consideration. In that connection the purpose of the present paper is to investigate the economic value of culling information for situations with and without quotas. This value is very important because it sets an upper limit on the costs of the very decision support system. In a later paper various methods for ranking of cows for replacement will be evaluated.

In order to determine the value of culling information the net returns to housing, labour and management were calculated using three different replacement policies in situations with a milk quota as well as situations without a milk quota.

Similar studies have been carried out under American and Dutch conditions by Marsh et al. (1987) and Dijkhuizen & Stelwagen (1988), respectively. However, neither of those have considered situations with a milk quota. Another difference from the present study is that the works mentioned are based on stochastic simulation. In the present study the net returns are calculated analytically from a stochastic replacement model based on dynamic programming.

#### *Theoretical aspects of a milk quota*

Kristensen (1989) found that the variation in future profitability due to variation in milk yield and calving interval among individual cows is considerably smaller under a milk quota than in situations without a quota. (The future profitability is defined as the gain—positive or negative—from keeping the cow for at least one additional period instead of replacing it immediately).

These results indicate that the economic value of information on milk yield and calving interval should be less under milk quotas. By relatively simple mathematical arguments it can be shown that the reason is the very nature of the production quota. The net returns ( $R$ ) per cow per year as a function of milk yield ( $Y$ ) per cow per year can be split up in a "constant" part  $c$ , which is independent of milk yield, and a part which is fairly proportional to milk yield (except for a decreasing gross feed efficiency). Thus we have

$$R = c + (p_m - 0.4p_f/e) Y \quad (1)$$

where  $p_m$  is the milk price,  $p_f$  is the price of a Scandinavian Feed Unit (SFU) and  $e$  is the gross feed efficiency. The constant 0.4 is the theoretical energy requirement in SFU to produce 1 kg fat corrected milk (4%). For convenience we shall assume  $e$  to be constant.

In a situation without a milk quota, the total net returns ( $H$ ) of the herd are set by the number of cows ( $N$ ):

$$H = RN = cN + (p_m - 0.4p_f/e) YN. \quad (2)$$

If a milk quota of  $M$  kg 4% milk is introduced the number of cows and the average milk yield must be adjusted to meet the quota. Thus the total net revenue  $H$  is set by the size of the milk quota:

$$H = R(M/Y) = cM/Y + (p_m - 0.4p_f/e) M. \quad (3)$$

A higher value of a high yielding cow compared to a low yielding cow is due to a positive marginal net return from increased milk yield. Therefore it is relevant to investigate the marginal value of larger milk yield per cow. From Eq. 2 we get

$$dH/dY = (p_m - 0.4p_f/e) N. \quad (4)$$

Under the assumptions made, the marginal value is seen to be constantly equal to the gross margin per kg of milk, which under usual conditions is positive. In the quota situation we get from Eq. 3:

$$dH/dY = -cM/Y^2. \quad (5)$$

The constant  $c$  contains the costs, that are directly related to the number of cows (the costs of keeping a cow). It can be calculated as the value of calf and weight gain less feed costs for maintenance, gain and embryo, and other costs that do not depend on yield. Thus

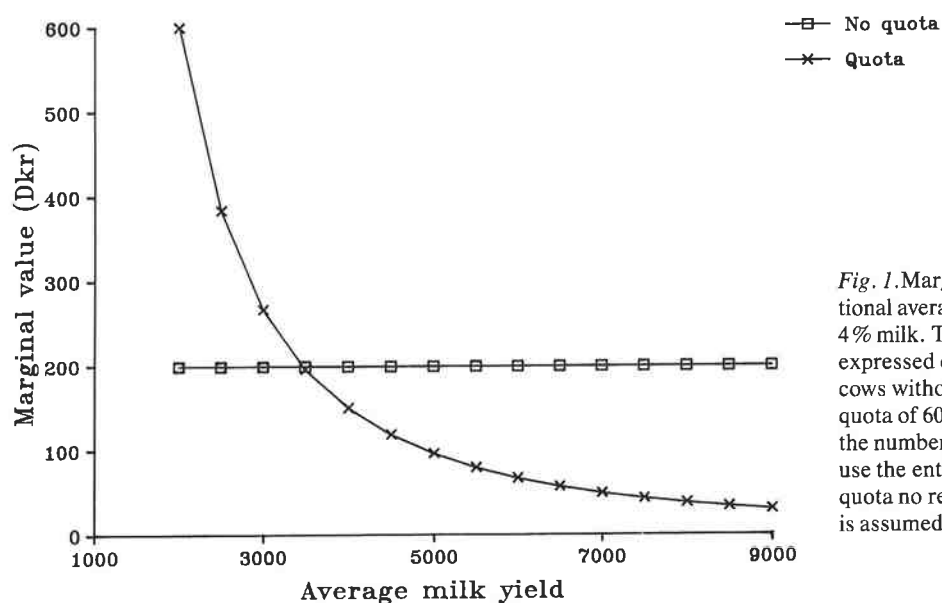


Fig. 1. Marginal value of an additional average milk yield of 1 kg 4% milk. The marginal value is expressed on herd basis with 100 cows without milk quota or a quota of 600 000 kg 4% milk and the number of cows needed to use the entire quota. Under a quota no restriction on herd size is assumed.

under usual conditions  $c$  will be negative, which means that the marginal value also in this case is positive, but unlike Eq. 4 it is *decreasing by the square of milk yield*.

Realistic values of  $p_m$ ,  $p_f$ ,  $e$  and  $c$  could be 2.70 Dkr, 1.60 Dkr, 0.9 and -4000 Dkr. respectively. If these values are used in Eqs. (4) and (5) we find from (4) that the marginal value is 1.99 N without a quota and 4000  $M/Y^2$  under a quota. If we assume the housing capacity to be  $N=100$  cows (no quota) and the milk quota to be  $M=600\,000$  kg milk (no restriction on herd size) the marginal value of 1 kg additional average milk yield of the herd is as shown in Fig. 1.

It should be emphasized that the two curves of the figure do not apply to the *same* herd. Like Eqs. (4) and (5) they reflect the *pure* effects of limits in *either* herd size *or* milk production. The situation of a herd not using a quota entirely is not covered by this very simple model. However, the parameter values used in the figure have been chosen in order to make the two curves comparable at a realistic level of milk yield (from 6 000 kg milk and up).

As it appears from Fig. 1, the marginal value of increased milk yield for any realistic level of milk yield is much smaller under milk quota than without a quota. At an average milk yield of 7 000 kg the marginal value is more than 4 times larger without a quota. In other words the economic advantage of increasing the average milk yield of the herd through replacement is much smaller in a situation with a quota than without a quota where the housing capacity is the major limitation.

Also the costs of keeping a cow (expressed by  $c$ ) should be considered. Under a quota, the marginal value in Eq. (5) is proportional to  $c$ . Large costs of keeping a cow (i.e.  $c$  numerically large) mean larger marginal value and consequently larger propensity to replace because of low milk yield. Without a quota the situation is different, since  $c$  does not influence the marginal value at all, as it is seen from Eq. (4).

The benefits of decreasing the average costs of keeping a cow by means of replacement is not larger under a quota than without a quota, as the number of cows is smaller rather than larger. Relatively, however, decreasing costs becomes more interesting under milk quota, because the possibilities to increase herd returns by higher milk yield are vanishing.

The difference in the marginal value of increased milk yield per cow may also be

explained by intuitive arguments. The optimization of replacement has as its basic purpose to ensure that at any time the most effective cows are used for production. Without a quota, where housing capacity is the major limitation, individual efficiency is equivalent to herd efficiency, because the number of cows is fixed. Under a (sufficiently low) quota the housing capacity is not limiting, which means that the efficiency of the individual animal is not necessarily equal to herd efficiency. If the quota is not met at the current efficiency there will always be the possibility to increase the number of cows by one or more animals. This is particularly relevant when the costs of keeping a cow ( $c$  in Eq. (5)) are low and/or the average milk yield is high (cf. Fig. 1).

## MATERIAL AND METHODS

As a consequence of the arguments of the previous section, we should expect the value of culling information, that primarily includes milk yield (directly and indirectly) to be considerably lower under a milk quota than without a quota. In order to test this hypothesis the net returns to housing, labour and management were calculated using 3 different replacement policies in situations with a milk quota as well as situations without a milk quota. The net returns are expressed relative to the most limiting factor, i.e. per kg milk under quota and per cow per year without a quota.

The three policies were formulated as follows: 1) Replace cows that fail to conceive within 154 days; 2) replace cows that fail to conceive within 238 days; and 3) replace cows according to an optimal policy from the stochastic replacement model used by Kristensen (1989).

Policy 1 and 2 were chosen because they match two of the policies tested in a similar study by Dijkhuizen & Stelwagen (1988) who compared the returns from 4 policies in situations without a milk quota. In this study Policies 1 and 2 are the same no matter if a milk quota is present or not. Policy 3, on the other hand, is different depending on the presence or absence of a milk quota. In each situation a policy is used that maximizes net return relative to the most limiting factor (i.e., per kg milk or per cow respectively). In the following an optimal policy in the absence of a quota is denoted as Policy 3a, and an optimal policy under a quota is denoted as Policy 3b.

Under Policy 1 and 2 cows were voluntarily replaced at 32 and 40 weeks after calving respectively. Under policy 3 cows were voluntarily replaced at the optimal stage of lactation depending on the individual properties of the cow.

In similar studies Marsh et al. (1987) as well as Dijkhuizen & Stelwagen (1988) have used stochastic simulation models for evaluation of policies. In this study a direct calculation of net returns is carried out. The calculations are based on the stochastic dynamic programming model used by Kristensen (1989), which may be regarded as a Markov decision process. The biological parameters of the model are published by Kristensen (1986). To describe the method used for calculation we shall introduce the notation of such process.

The system (a cow) is defined by its *state*  $i$  ( $i=1, \dots, I$ ) defined by the present properties (genetic class, lactation number and stage, milk yield in previous and present lactation and expected length of the calving interval). As soon as the state is observed we will have to choose an *action*  $d$  (in this case  $d=1, 2$  for "keep" or "replace"). A set of actions (one for each possible state) makes up a *policy*  $s$ . We shall denote as  $s(i)$  the action that the policy  $s$  defines for state  $i$ . Depending on the state and action, a *reward*  $r_i^d$  is gained (in this case the reward is the net return). Further we assume that some physical quantity denoted as  $m_i^d$  is involved. In this case the physical quantity may either be the amount of milk produced by a cow in state  $i$  when the action  $d$  is taken or it may be the duration of the present stage depending on whether we are producing under a quota or not. The *transition probability*

from state  $i$  at the present *stage* to state  $j$  at the next stage also depends on the action taken, and it is denoted  $p_{ij}^d$ . If  $d=s(i)$  the symbols  $r_i^d$ ,  $m_i^d$  and  $p_{ij}^d$  are also written as  $r_i^s$ ,  $m_i^s$  and  $p_{ij}^s$  respectively. It is possible to show that for any policy  $s$ , we have:

$$g^s m_i^s + f_i^s = r_i^s + \sum_{j=1}^I p_{ij}^s f_j^s, \quad i = 1, \dots, I, \quad (7)$$

where  $g^s$  is the average net return per unit of the physical quantity represented by  $m_i^s$  under the policy  $s$ . In the equations (7),  $f_1^s, \dots, f_I^s$  together with  $g^s$  must be regarded as unknowns. The variable  $f_i^s$  is called the relative value of state  $i$  under the policy  $s$ , but it is of no interest in this connection. Depending on whether  $m_i^s$  is the milk yield or the duration of the stage,  $g^s$  gives the average net return per kg milk produced or per unit of time (i.e. per cow per year). The  $I$  equations of (7) may be solved for the unknowns  $g^s$  and  $f_1^s, \dots, f_I^s$  if we add the restriction  $f_I^s = 0$ . Thus for any policy we can calculate the net revenue per cow per year and per kg of milk produced by solving a set of  $I+1$  simultaneous linear equations.

The model used contains approximately 180 000 states, so in principle a set of 180 000 linear equations should be solved involving the inversion of a matrix of the dimension  $180\,000 \times 180\,000$ . However by using a technique called hierarchic Markov processes developed by Kristensen (1988; 1989) the number of equations to be solved are reduced to 6.

The direct calculation of net returns has the advantage over stochastic simulation by random number generation that it reduces the calculations drastically and avoids the problem of variation in simulation results without reducing the stochastic elements in any way. Differences between returns from policies are therefore absolutely precise and not influenced by random variation.

The prices and other conditions used in the calculations are shown in Table 1.

## RESULTS AND DISCUSSION

The technical and economic impacts of the replacement policies are shown in Table 2. Policy 3a, which is optimal in the absence of a quota, is characterized by a very intensive culling for milk yield and reproductive performance. Thus the average milk yield is 6.6% higher than under Policy 2 where no culling for milk yield and almost no culling for length of calving interval takes place. The more intensive culling under Policy 3a is also illustrated by the fact that the average stage of lactation for replacement (voluntary and involuntary) is 7 weeks earlier than under Policy 2.

Table 1. *Conditions used in the calculation of net returns*

Prices (Dkr)	
Milk (kg FCM)	2.40
Basic food (SFU)	1.30
Food for milk production (SFU)	1.45
Calf	1 400.00
Heifer	9 000.00
Young cow (kg live weight)	11.50
Older cow (kg live weight)	11.00
Interest rate (corrected for tax and inflation, %)	3
Herd level of milk yield (week 1–40, 1st lact.)	5 800.00

Policy 3 b, which is optimal under a quota, is in fact not very different from Policy 2. The average milk yield is only 1.4% higher and the average stage of lactation for replacement is only 3 weeks earlier than under Policy 2. Thus the culling for milk yield and reproductive performance is much less intensive under Policy 3 b compared to 3 a.

If we turn to the economic results, it appears that in the situations without a milk quota Policy 3 a is evidently better than the others. The net returns to housing, labour and management are 3.3% higher than for Policy 1, which ranks second. The results in absolute figures are not directly comparable to those of Dijkhuizen & Stelwagen (1988) because they included housing costs in the calculations. Since, however, the housing costs are independent of the replacement policy, the *differences* between net returns from different policies may be compared.

In this study the difference between Policy 3 and Policy 1 equals 308 Dkr per cow per year. The corresponding difference in the study of Dijkhuizen & Stelwagen was 33 Dfl. or approximately 125 Dkr per cow per year (under "average" level of reproductive performance). Thus the benefits from policy 3 seem to be larger under Danish conditions than under Dutch in the absence of a milk quota. The reason for this is probably, that under Danish conditions the optimal replacement rate (as well as observed rates) is considerably higher than in most other countries due to a relatively high value of culled cows.

Under a quota the impression is fundamentally different. Naturally Policy 3 b is the most profitable, but the benefits compared to Policy 2 are less than 1% in net returns to housing, labour and management. It is therefore evident that the costs of a decision support system for culling must be very low in order to be profitable. A rough estimate of the maximum costs is approximately 45 Dkr per cow per year.

Policies 1 and 2 primarily affects herd profitability by preventing low milk yield per cow per year to result from long calving intervals. In policy 3 a and 3 b, the yield ability of individual cows is also taken into account, which leads to a more efficient selection. But with a smaller marginal value of average milk yield under quota, the benefits become smaller.

As it appears from Table 2 the mutual ranking of Policies 1 and 2 depends on the

Table 2. *Technical and economic impact of different replacement policies*

	Policy			
	1	2	3 a <sup>a</sup>	3 b <sup>a</sup>
Milk yield, kg/cow/year	7 082	6 896	7 350	6 991
Average time for replacement, weeks after calving <sup>b</sup>	25	28	21	25
Annual replacement rate	50	35	59	38
Net returns to housing, labour and management				
Dkr/cow/year, abs.	9 236	9 150	9 544	(9 319)
Dkr/cow/year, rel.	100.0	99.1	103.3	(100.9)
Net returns to housing, labour and management				
Dkr per 1 000 kg milk, abs.	1 304	1 327	(1 299)	1 333
Dkr per 1 000 kg milk, rel.	100.0	101.8	(99.6)	102.2
Number of cows assuming quota, relatively	100.0	102.7	96.4	101.3

<sup>a</sup> Policy 3 a is an optimal policy without a quota, and policy 3 b is optimal under a quota.

<sup>b</sup> Includes involuntary replacements.

presence or absence of a milk quota. The reason for this is intimated by the technical results which show that Policy 1 in its consequences is quite similar to Policy 3 a, and Policy 2 is quite similar to Policy 3 b.

The net returns per 1000 kg milk under policy 3 a is 1299 Dkr. This amount is the expected net revenue to be gained if the optimal policy from a situation without a milk quota is used under a quota. It should be noticed that under a quota the returns from policy 3 a are lower than even those from Policy 1 and considerably lower than those from Policy 2. These results show that a decision support system based on maximization of net revenue per cow will directly misinform the dairy farmer if used under a quota.

## CONCLUSIONS

In the absence of a milk quota there are considerable benefits from using an efficient selection of cows with the highest expected milk yield, but under a quota the benefits are negligible if compared to the very simple policy of only replacing cows which fail to conceive within 238 days. If a system based on calculations assuming no quota is used under a quota the dairy farmer will be directly misinformed. Decision support systems for culling should be specifically designed for the quota situation, where reductions of costs are the most important means for improving herd net returns. Emphasis should therefore be put on information that support reductions on average costs of keeping a cow by replacing, for example, cows with a high risk of diseases.

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# Ranking of Dairy Cows for Replacement

## *Alternative Methods Tested by Stochastic Simulation*

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The preferred tool in theoretical dairy cow replacement studies has been stochastic dynamic programming. Under practical conditions, however, this method is less suitable due to computational complexity. In order to develop a ranking criterion for practical use, two criteria were tested by means of stochastic simulation based on random number generation. Under the first criterion the ranking was provided by a dynamic programming model, but instead of herd individual conditions a set of standard conditions was used. Under the second criterion the cows were ranked according to their expected net returns to housing, labour and management during the next 12 months. Both criteria were tested in the presence and absence of a milk quota. The simulation results showed that both criteria are suitable for practical implementation in both situations. Thus the final choice depends on other considerations including implementation and operation costs as well as comprehensibility. A third criterion called expected maximum net returns, which has been suggested in the literature, was considered, but, rejected from theoretical and empirical reasons. *Key words:* culling information, decision support, milk quota.

## INTRODUCTION

The dairy cow replacement problem has been the object of several studies in the literature. The preferred tool for optimization has been stochastic dynamic programming, and very detailed models have been developed by van Arendonk (1985, 1986), van Arendonk & Dijkhuizen (1985) and Kristensen (1987, 1989). The works mentioned have resulted in profound knowledge of the nature of the problem as well as the factors influencing the optimal replacement policy. Further Kristensen (1989) has discussed the problem of adjusting such models to fit a situation with a milk quota.

Thus from a theoretical point of view the replacement problem should be regarded as solved to a satisfactory degree based on detailed models. These models, however, are at present not suitable for practical use in commercial dairy herds because of the extreme amount of computing time required to get an optimal policy. Therefore alternative approaches must be considered to provide the dairy farmer with culling information. In a paper of van Arendonk (1988) one method has been described, but the situation with a milk quota has not been considered, and the benefits from the method has not been compared to those of other alternatives.

In Denmark a decision support system for culling of dairy cows in commercial dairy herds is under consideration. In that connection the purpose of the present paper is to evaluate different ranking criteria of cows in commercial dairy herds in the presence and absence of a milk quota. The evaluation will be based on the total net returns to housing, labour and management from all dairy cows of the herd when a certain ranking criterion is

used. The net returns of the herd are calculated by means of a stochastic simulation model. In a previous paper (Kristensen & Thysen, 1990) the economic value of culling information has been studied.

### THE SIMULATION MODEL

A model consisting of dairy cows and the associated female offspring was used. Each cow and heifer is in the model represented by its *state* defined by the values of a number of *state variables* representing the most important characteristics of the animal. Transitions from one state to another are possible at regular 4 week intervals (*stages*). The state of each animal at the next stage is determined by drawing a random number from the relevant probability distribution defined by the present state (and whether the present animal is kept or not). The main characteristics of the model are summarized in Table 1. A fixed maximum herd size must be defined by the user.

The distributions and mutual relations of the state variables of cows have been described and estimated by Kristensen (1986). The milk yields are corrected for herd level, breeding value of father and length of the calving interval. Cows may be replaced voluntarily or involuntarily. The involuntary replacements caused by death or bad health are governed by probabilities depending on lactation number and stage of lactation. Furthermore, cows that fail to conceive within 238 days after calving are always replaced (at 280 days after calving).

The voluntary replacement policy is represented by a set of numbers defining the ranking in descending order of the possible states in the model according to alternative estimates of future profitability as defined in a later section. The replacement decision is modelled as follows: At each time stage cows after involuntary cullings and heifers ready for calving are counted. If the number of cows is less than maximum herd size, the

Table 1. *Main characteristics of the simulation model*

<i>Cows</i>	
Number of 4 week stages per lactation	11–18 <sup>a</sup>
Maximum number of lactations per cow	6
Replacement if not pregnant before	238 days
Stochastic state variables	
Breeding value (milk yield) of father	5 classes
Milk yield, previous lactation	15 classes
Milk yield, present lactation	15 classes
Length of calving interval	8 classes
Deterministic state variables	
Lactation stage	18 classes
Lactation number	6 classes
Total number of states (approximately)	180 000
Probability of a calving to result in a surviving heifer	0.45
<i>Heifers</i>	
State variable	
Age or reproductive status	41 classes
Minimum age of 1st breeding	56 weeks
Probability of conception per stage	0.33
Age of disposal of open heifers	116 weeks

<sup>a</sup> Depending on the length of the calving interval.

available heifers are used for increasing the herd size to maximum. The remaining heifers (if any) are used for voluntary replacements of the lowest ranking cows as long as the ranking number is below a limit defined by the user. In the simulation model the user may allow for buying heifers at the market, but in the presented study, only heifers raised in the same herd are used for replacements.

At the beginning of each stage, the milk yield and net returns (to housing, labour and management) of each individual cow are calculated from the state. Female calves are transferred to heifers and heifers are transferred to cows at market prices. The costs of raising heifers are not included.

Heifers are only represented by their age and/or pregnancy status. Heat detection and insemination of heifers is initiated at the age of 56 weeks, and a heifer is culled at the age of 116 weeks if still not pregnant.

When the simulation model is used in a quota situation, the ranking numbers are calculated in a relevant way (i.e., aiming at maximum net returns per kg milk). The quota as such is not included, and followingly short time adjustments of herd size are not considered. Since a fixed herd size is assumed, the simulation results should be interpreted as the net returns per kg milk being gained in the long run under the ranking criterion. In practise the number of cows should be adjusted afterwards to meet the quota by the achieved milk yield.

In all cases a dairy herd of 100 cows was simulated over a period of 100 years in order to decrease the random variation on results.

## RANKING CRITERIA

### *Dynamic programming with standard conditions*

Prices and other conditions vary from herd to herd and therefore the optimal policy of one herd is not necessarily optimal in another. In principle it would be necessary to calculate an optimal replacement policy of each individual herd. As mentioned in the introduction a direct implementation of a detailed dynamic programming model is not suitable at present because of the extremely comprehensive calculations involved. Furthermore, precise information on prices, especially feed prices, are difficult to obtain.

However, Kristensen & Østergaard (1982) have shown that the mutual ranking of cows is very stable under variations in prices and herd level of milk yield. Thus it might be relevant to calculate an optimal policy and an optimal mutual ranking of states under some standard conditions and use this ranking in all herds. This method has been used in The Netherlands (van Arendonk, 1988). We shall refer to this criterion as dynamic programming with standard conditions (*DPS*).

Table 2. *Standard prices and herd level of milk yield*

Prices (Dkr)	
Milk (kg FCM)	2.40
Basic feed (SFU)	1.30
Feed for milk production (SFU)	1.45
Calf	1 400.00
Heifer	9 000.00
Young cow (kg live weight)	11.50
Older cow (kg live weight)	11.00
Interest rate (corrected for tax and inflation, %)	3
Herd level of milk yield (week 1–40, 1st lact.)	5 800.00

The standard conditions used here are presented in Table 2. By use of a random number generator, 100 sets of alternative conditions have been generated. Denote as  $f^0$  a vector with 7 elements representing the prices (excluding the price of older cows) and the herd level of milk yield in Table 2. An alternative set of conditions  $f^i$  ( $i=1, \dots, 100$ ) is generated as follows:

$$f^i = f^0 + 0.15 r^i \cdot f^0, \quad i=1, \dots, 100, \quad (1)$$

where  $r^i$  is a diagonal matrix of which the diagonal elements form an independent random sample drawn from an equal distribution over the interval  $[-1; 1]$ . Eq. (1) expresses that in each set of conditions the individual prices and level of milk yield are varied independently over an interval defined from the original value of Table 2 plus or minus 15%. The price of older cows is linked to the price of young cows in such a way that the price *difference* varies in a similar interval.

Several ranking criteria involving future profitability can be derived from a dynamic programming model. Three alternative definitions of future profitability were tested:

- A. The advantage of keeping the cow for at least 4 weeks compared to immediate replacement.
- B. The advantage of keeping the cow until it is replaced involuntarily or because of age compared to immediate replacement.
- C. The advantage of keeping the cow at least until the next calving compared to immediate replacement.

In all cases the future profitability may be positive or negative.

For each of the 100 sets of conditions optimal policies and ranking according to A, B and C were calculated in the presence and absence of a milk quota using the optimization model described by Kristensen (1988; 1989; 1990). The economic results of the three definitions were compared, and the best one was chosen for comparison with ranking of cows under standard conditions.

#### *Expected net returns during the next 12 months*

A very simple ranking criterion is the expected net returns during the next 12 months (ENR). In the absence of a quota we use the expected net returns per cow, and under a quota we use the expected net returns per 1000 kg milk. A period of 12 months is used because it usually contains the next calving.

The ENR is calculated by means of stochastic simulation. In each of the 180 000 states 13 additional 4 week stages are simulated a number of times. For each replication the ENR is calculated, and the final ranking is defined by the average ENR over replications. If a cow is involuntarily replaced in a replication the results of the new heifer are used in the remaining period.

#### *Expected maximum net returns*

Based on works of Kuipers (1982) and Congleton et al. (1988) a third criterion called expected maximum net returns (EMNR) has been considered. The idea is to calculate the expected average net returns per cow per stage (or per 1000 kg milk under a quota) from the present stage and 1, 2, 3, ... stages ahead until the maximum age is reached. The EMNR is defined as the largest of these calculated averages.

Unlike the ENR criterion, the EMNR is *not* calculated over a fixed period ahead, but over the period (short or long) that maximizes the expected average returns. In other words it is the expected average returns until a cow identical in all respects (including *present* age) to the present one profitably could replace it. On the other hand it is *not* the

period until the optimal replacement time. If, for instance, a cow has got a very high EMNR, the optimal replacement time may very well be *later* than the time that gives the highest EMNR. In that case the new cow is typically less profitable than the present one, and thus it would be profitable to keep for a longer period.

Because of these arguments, which were confirmed by empirical simulation results, the EMNR criterion was rejected.

## RESULTS

### *Ranking on future profitabilities*

In Table 3 the simulation results using ranking on future profitabilities defined according to A, B and C are compared in the absence as well as the presence of a milk quota. Furthermore, the simulation results are compared to those under optimal policies without any restrictions concerning heifers.

In the absence of a quota the results are presented as means and percentiles of net returns *per cow per year* to housing, labour and management. In this situation it is assumed that only home raised heifers enter the herd, and that if a heifer is available it will always replace the lowest ranking cow (in other words *all* pregnant heifers will enter the herd as it has been common practise in Denmark).

In the presence of a quota the results are presented as means and percentiles of net returns *per 1 000 kg milk* to housing, labour and management. In this situation the optimal replacement rate is considerably lower than without a quota as reported by Kristensen (1989). Thus the assumption that all heifers must enter the herd has to be relaxed. Therefore heifers only entered the herd if the future profitability of the lowest ranking cow was negative.

It appears from Table 3 that the 100 sets of conditions represent widely different economic results. Thus under the optimal policies the net returns vary from 5 128 to 14 665 Dkr per cow per year in the absence of a quota, and from 844 to 1 804 Dkr per 1 000 kg milk under quotation.

From the results, we find that in the absence of a quota the future profitability defined as C is superior to A and B both when evaluated on means and extremes. Under a quota,

Table 3. Simulation results using future profitabilities defined in three alternative ways (see the text for definitions) compared to results from optimal policies without any restrictions on number of heifers

Per- cen- tile <sup>a</sup>	Net returns/cow/year				Net returns/1 000 kg milk			
	Optimal policy	Definition <sup>b</sup>			Optimal policy	Definition <sup>b</sup>		
		A	B	C		A	B	C
1	5 128	-1 442	-1 188	-1 048	844	-10	-15	-13
5	6 553	-704	-708	-446	924	-4	-10	-9
50	9 402	-141	-168	-138	1 309	1	-2	-4
95	13 014	-54	-81	-60	1 724	6	5	2
100	14 665	-8	-40	-27	1 804	8	7	4
Mean	9 540	-233	-227	-185	1 323	1	-2	-4

<sup>a</sup> Calculated for each variable independently of the others.

<sup>b</sup> Deviations from optimal policy without restrictions.

however, it appears that the future profitability should be defined as in A, but the differences among definitions are small. Under definition A the economic result is even 1 Dkr better than under the optimal policy, but no significance should be attached to that.

#### *Ranking on the DPS criterion*

In Table 4 the simulation results using ranking on future profitabilities calculated under standard conditions are shown. In the quota situation the future profitabilities are calculated according to Definition A, and in the absence of a quota Definition C is applied.

In the situation with no milk quota we find that the economic result under standard ranking is even better than under the herd individual ranking: The difference, however, is only 20 Dkr per cow per year, and in 98 cases out of 100 the numerical difference is less than 100 Dkr. If we assume the differences to form an independent sample from a normal distribution, the 20 Dkr superiority is significant at the 0.1% level. Thus the difference seems to be real, even though the economic impact is negligible. The explanation seems to be that the optimal replacement rate under the standard conditions just happens to be the same as the one achieved when all heifers are raised and all of them enter the herd at calving.

We can conclude, that the DPS criterion (with the future profitability defined according to C) seems to be suitable for practical implementation in the absence of a milk quota. Further the standard conditions should be chosen so that they fit the replacement rate wanted.

Under a milk quota it appears from Table 4, that *on average* the standard ranking has done quite well (only 2 Dkr below the optimal policy), but from the percentiles we see that in some cases the loss compared to the optimal policy is rather large (up to 30 Dkr). Therefore an adjustment of future profitabilities was considered. Define the *replacement costs* as

$$c = c_h - 480c_y - c_c, \quad (2)$$

Table 4. Simulation results using future profitabilities calculated under standard conditions (the DPS criterion) compared to the relevant results under herd individual conditions from Table 3

Per- cen- tile <sup>a</sup>	Net returns/cow/year		Net returns/1 000 kg milk		
	Individual conditions	DPS cri- terion <sup>b</sup>	Individual conditions	DPS criterion <sup>a</sup>	
				Uncor- rected	Corrected <sup>c</sup>
1	-1 048	-914	-10	-30	-10
5	-446	-411	-4	-13	-5
50	-138	-124	1	0	1
95	-60	-59	6	6	5
100	-27	-29	8	9	7
Mean	-185	-165	1	-2	0

<sup>a</sup> Calculated for each variable independently of the others.

<sup>b</sup> Deviations from optimal policy without restrictions.

<sup>c</sup> The future profitability corrected according to Eq. (3).

where  $c_h$  is the price of a heifer,  $c_y$  is the price per kg live weight of a young cow and  $c_c$  is the price of a calf.

From the individual results under a milk quota it appeared that the standard ranking was good when the replacement costs under the individual conditions were close to those under standard conditions, but deviations increased when the numerical differences in replacement costs increased. Therefore a correction of future profitabilities should be related to the replacement costs. Several corrections were tested, but a good fit was achieved by the following one:

$$f_i^* = f_i + (c - c_s)^3 / 1200^2, \quad (3)$$

where  $f_i^*$  and  $f_i$  are the future profitabilities of state  $i$  (corrected and uncorrected, respectively),  $c$  is the replacement costs under individual conditions and  $c_s$  is the replacement costs under standard conditions.

The last column in Table 4 summarizes the results under corrected standard ranking under a milk quota. It appears that when the corrections are carried out, the standard ranking is just as good as the individual ranking (definition A). Thus we can conclude, that the DPS criterion (with future profitability defined according to A and corrected as in Eq. (3)) is also suitable under a milk quota.

#### Ranking on the ENR criterion

Under the ENR criterion the standard conditions of Table 2 were used. Some experimentation with the number of replications in the calculation of ENR was done in the situation without a quota. The results from the simulations are shown in Table 5. It appears that the economic consequences depend very much on the number of replications, but from the trend we can conclude that the result, where ENR is calculated over 100 replications must be very close to the best possible under this criterion. The difference of 68 Dkr from the result under an optimal policy clearly illustrates that also this criterion is relevant for practical implementation in the absence of a quota.

Under a milk quota the ENR criterion was only tested with ENR calculated over 100 replications, since that number showed to be close to the ideal situation in the absence of a

Table 5. Net returns (Dkr per cow per year and per 1 000 kg milk) to housing, labour and management under the ENR criterion compared to an optimal policy without restrictions concerning heifers

Number of replications in calculating ENR	Deviation from optimal policy	
	Dkr/cow/year <sup>a</sup>	Dkr/1 000 kg milk <sup>b</sup>
1	-574	-
10	-224	-
25	-125	-
50	-90	-
100	-68	-3
Optimal policy (abs. fig.)	9 544	1 333

<sup>a</sup> Average of 2 simulation runs over 100 years.

<sup>b</sup> Average of 10 simulation runs over 100 years.

quota. The result (Table 5) shows that also under a quota this criterion is very relevant, since the difference from the optimal policy is only 3 Dkr per 1000 kg milk.

## DISCUSSION AND FINAL CONCLUSIONS

Both criteria tested have proved to be suitable for practical implementation in the presence as well as in the absence of a milk quota, and the economic results under the two criteria differ only insignificantly. Thus the final decision should be based on other considerations than those presented in the previous sections.

From the test of the ENR criterion it is clear that ENR must be calculated very precisely in order to give satisfactory results. The method used for calculation of ENR in this study was stochastic simulation, which is very time consuming and therefore expensive. In an implementation more simple deterministic methods should be considered, but it is very important that the precision is not relaxed. That applies particularly in the quota situation, where the value of culling information is considerably lower than without quotas as reported by Kristensen & Thysen (1990).

On the other hand the ENR criterion has the advantage over the DPS, that it is very easy to understand, whereas the future profitabilities used under DPS are rather abstract. Furthermore the ENR criterion may easily be adjusted to include informations not considered in this study (e.g. health data). If such information should be included under the DPS criterion a completely new and very complicated replacement model had to be constructed.

The final choice of criterion should be based on an analysis covering all aspects including the costs of implementation and operation. In such analysis the results of this study should be considered, but they only form one element of the problem.

If the DPS criterion is used without a milk quota, the future profitability should be defined as the benefits from keeping the cow at least until next calving compared to immediate replacement. Further it is important that the standard conditions give rise to an optimal replacement rate close to the one desired.

Under a quota the future profitability should be the benefits from keeping at least for one additional stage (4 weeks) compared to immediate replacement. The future profitabilities should in this situation be corrected according to Eq. (3).

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# Conclusions and outlook

In the introduction, the main purpose of this thesis was defined as the adaptation of Markov decision programming techniques in order to be able to cope with the animal replacement problem in a satisfactory way. The problems were identified as the *dimensionality problem*, the *uniformity problem* and possible *herd restraints*. It is now the time and place to conclude to what extent the problems have been solved.

We can conclude from the preceding chapters that the problems are mutually inter-dependent. As shown in Chapter VII, the solution of the uniformity problem may also in some cases contribute to the solution of the dimensionality problem by state space reduction, and as it appears from Chapter VIII, the introduction of herd restraints makes the dimensionality problem worse because such models inevitable become very large when all animals must be considered simultaneously. Thus a true solution to the herd restraint problem will also in some models contribute to the solution of the dimensionality problem.

The main contribution to the solution of the dimensionality problem is, however, the formulation of the hierarchic Markov process. The computational advantages of the method have been illustrated by theoretical considerations and a numerical example in Chapter II. The major forces of the technique may be summarized as follows:

- 1) The method is exact.
- 2) The method is much faster than the value iteration method (i.e. the usual dynamic programming technique) as it appears from Chapter II.
- 3) The method makes it possible to calculate the consequences of an arbitrary policy directly as it appears from Chapter X. Furthermore it is possible directly to calculate many technical and economic results under a given policy as discussed in several chapters (e.g. annual replacement rate, annual milk yield, average litter size etc.)

Since the method is faster than the usual techniques it *contributes* significantly to the solution of the dimensionality problem, because the optimization of larger models becomes realistic in real time. On the other hand, the dimensionality problem has not been *solved*. The limits of the possible have been raised, but not removed.

As concerns the uniformity problem, the Bayesian updating technique of Chapter VII has been proposed. The force of the Bayesian approach is that it constitutes a framework for dealing with imperfect knowledge. At any level of knowledge it is possible to take an optimal action *under the present circumstances* (with imperfect knowledge). Thus the Bayesian approach seems more to be in accordance with a real life situation. It is expected that the method in particular will be relevant in dealing with categorical effects (such as diseases) as sketched in Chapter VII. However, that perspective has to be tested on real data before any finite conclusion may be drawn. As concerns the application to usual quantitative traits as milk yield or litter size the method is considered to be a significant contribution to the solution of the uniformity problem, because it makes it possible to distinguish the variation caused by different effects and thus explains the very nature of the variation in traits. Furthermore the technique may in some cases reduce the size of the state space without loss of information.

Two different herd restraints have been considered in this thesis. One restraint is a milk quota in dairy cattle, which has been dealt with by the introduction of an alternative criterion of optimality maximizing average net returns per unit of milk produced (Chapters V and VI). The new criterion solves the long term ranking problem of cows in a herd producing under a quota, but it does *not* solve the short term adjustment problem of deciding how many cows to use in the production at any time. To include this feature we will have to turn to the parameter iteration method applied in Chapter VIII to an other herd restraint which was a limited supply of heifers. It has been demonstrated that the

technique is superior to the usual single-animal models in a situation with shortage of heifers, but it is not an all-embracing method that may be transformed directly to all kinds of herd restraints. The basic idea is to take advantage of the fact that an optimal solution to the unrestricted problem is known and then consider in what way the restraint influences the total present/relative value of the herd. This *basic* idea may also apply to other restraints, but in each case some hard work is left determining the appropriate functions and parameters involved. Thus the herd restraint has not been *solved*, but the *framework* of an approximate method has been constructed.

A secondary purpose of the thesis was defined as illustration and discussion of the applicational perspectives of the techniques. In Chapter IX the applicational areas of the techniques were identified as *research, development of methods to be used in practice* and *direct application in commercial herds*. As illustrated in Chapters VI and X, we may conclude that the techniques are useful tools in research where the conditions and traits that influence the optimal policies may be studied. Furthermore they may be used in comparative studies in the development of operational methods to be used in commercial herds (cf. Chapter XI). As concerns direct application in such herds, we must expect that it will become technically possible, but whether or not it is appropriate will depend on the outcome of such studies, and no final conclusion may be drawn at present.

With regard to the need for future research activities, the solution of the dimensionality problem should be given high priority. This may be done in two different ways. One way is to raise the upper limit concerning the size of state spaces to handle. The day when a herd model as the one described in Chapter VIII may be solved exactly is, however, very far away, unless a real epoch-making discovery is done. Furthermore it is a question whether not the limits of the human mind regarding the ability to grasp such a model will be reached before. Another way to solve the problem would be to consider some kind of state space transformation which will reduce the state space with no (or at least little) loss of information. A further development of the Bayesian techniques might be a way of doing this, but much further research in this area is necessary.

Also the question of how to deal with traits re-

lated to the health of the animal need to be studied. Again the Bayesian technique might be a relevant tool. It seems obvious that the question of herd restraints must be a main task in the future, since in the real world animal production is performed in herds. Therefore the idea of a simple comparison of the animal in production with a possible replacement is not valid if all animals compete for the same scarce resource or production quota.

These future research areas are reminders demonstrating that further methodological research remains in order to fulfill the objective of this thesis completely, but it should be noticed that *even* if all methodological problems were solved and, accordingly, the rather technical objective was fulfilled, only part of the job had been done. It seems relevant also to consider the work in a wider perspective. On the long view, it is of course the idea that the developed techniques should support the farmer's decisions on which animals to replace. In other words, the techniques are intended to form the central element of a decision support system.

None of the techniques presented have been implemented in a decision support system, and it is natural to ask why not. The situation is in no way unique. Despite at least two decades of research in optimization and simulation models, very little has reached the farmer in the form of working decision support systems. The 3rd International Congress for Computer Technology in 1990 was devoted to "Successful Practical Applications". Afterwards a participant ironically said that some of the applications presented at the congress were "successful" while others were "practical" and some were just "applications". In other words, successful practical applications seem to be exceptions. The title of the 4th congress in the same series "Farm Computer Technology in Search of Users?" suggests that this is a general feeling.

The lack of successful decision support systems in animal production makes it relevant to consider in general what the success of a decision support system would depend on. In Figure 1 the basis of a decision support system is sketched. The elements of the figure should be interpreted as *necessary* conditions (or bottle-necks) that must be fulfilled. (In some cases they may even not be *sufficient*).

The necessary conditions are divided into 4 groups, depending on whether they are *hardware* related, *software* related, *method* related or *farmer* related. Some of the hardware conditions refer to

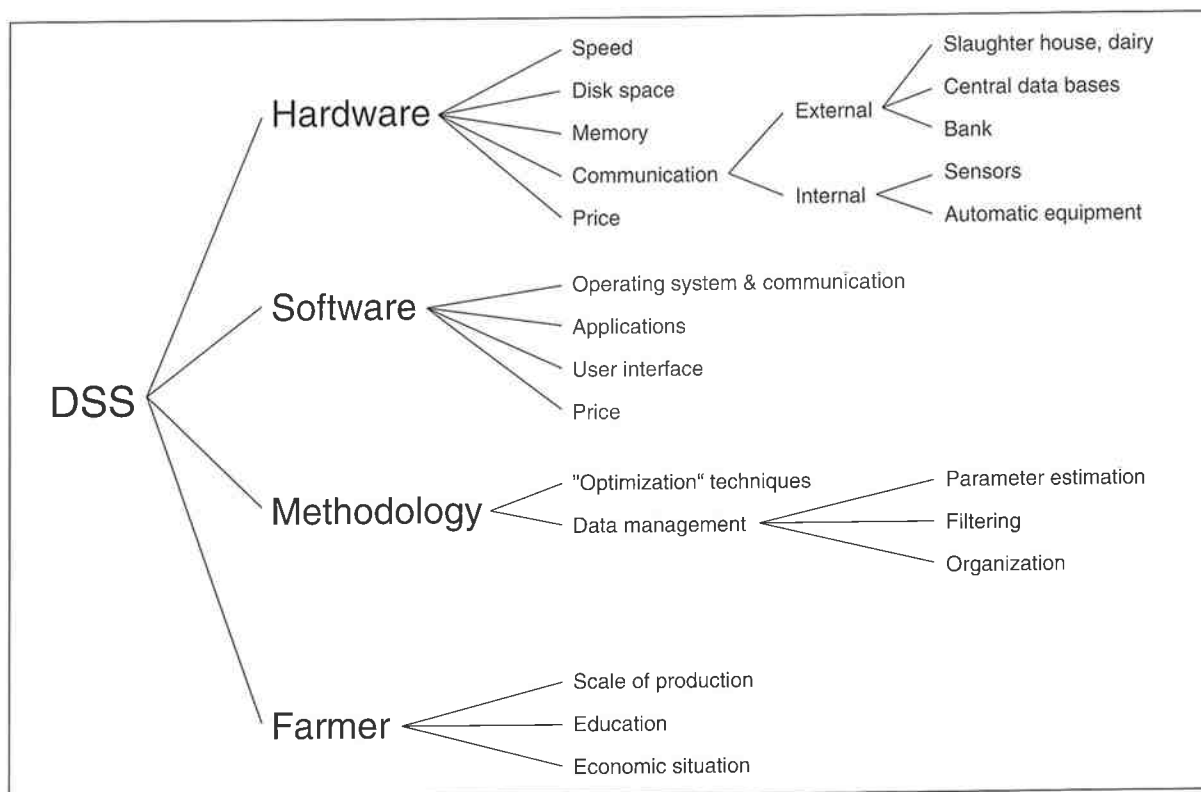


Figure 1. The basis of a decision support system (DSS).

performance characteristics like speed, disk space and internal memory. Some methods require even very powerful computers. In the *short run* the performance of computers may be a bottle-neck in the development of decision support systems, but, as illustrated in Chapter IX, the performance characteristics of computers improve at a very fast rate, so in the *long run* the performance of computers will hardly be a problem. The same applies to communication hardware making the computer able to receive data from external sources like classification of animals and milk from slaughterhouse and dairy, or from internal sources (sensors) like temperature and conductivity of milk or weight of animals. It may also be necessary to be able to send data from the computer to automatic equipment like milking robots or automatic concentrate feeding. The last condition mentioned in the figure is the price which has to be sufficiently low to make the investment in hardware profitable.

The *software* conditions refer partially to operating systems and communication software which, however, is commercially developed concurrently with hardware. The development of applications

(i.e. computer coding of optimization techniques and data management) with good user interfaces is more important in this context. Especially the user interface has often been overlooked, because most attention has been paid to the coding of the methods developed. Nevertheless, a decision support system with an improper user interface has no chance on the market. The best way to ensure a high level in software development (including user interfaces) is probably to realize that this part of the job should be handed over to specialists in programming (e.g. employed by the advisory service). The researcher, who developed the method, is usually not a programming expert, and furthermore he or she is certainly not impartial in the assessment of the applicational perspectives. On the other hand, a rapid implementation of new relevant methods demands a close cooperation between researchers, advisory service and education. Just as for hardware, the price of the software must be sufficiently low to make the investment profitable.

At least in research, most attention has been paid to *methodology*, in particular "optimization" techniques in a broad sense which in this connec-

tion also cover e.g. expert systems and simulation. Several methods and prototypes have been developed and praised by the researchers themselves, but as mentioned before, very little has reached the farm level. Since the utility value of the methods in several cases has been verified, we may conclude that lacking optimization techniques is hardly a bottle-neck in the spread of decision support systems. Just as important, but less studied, is data management. As a result of the technical progress the number of data sources is assumed to increase. Sensors and external communication lines are expected to provide huge amounts of registrations. The transformation of these registrations to data and information demands new methods for filtering, organization of data bases and estimation of parameters on herd level.

Last, but not least, the situation of the *farmer* must be considered. Is the production large enough to justify the purchase of a decision support system, and can he afford it? Even if the answer is yes, the education and lacking computer experience of the farmer may be a problem. This is probably the most limiting factor in the spreading of decision support systems. The lack of computer experience may to some extent be compensated by training and good user interfaces, but it should be recognized that it may take a generation before the average farmer is really familiar with computers and considers them as just as natural tools as pen and notebook. Also the educational level in a wider sense than computer experience is important. For instance, lack of knowledge on statistical theory may be a problem when dealing with stochastic modelling, confidence limits, sensitivity considerations etc.

This short general survey of conditions to be met in order to create a decision support system seems to show that it is not surprising that "successful practical applications" are very few. The main problem is that there has been extensive research in methodology, but the level of software development (in particular the design of user interfaces) and farmer education has not been raised simultaneously.

After these general considerations, focus will be turned back to the specific problem of integrating the methods presented in this thesis in a working decision support system. As all examples of the preceding chapters, this discussion will refer to a commercial dairy herd under Danish conditions.

The methods are very computer intensive, and already this fact creates problems in relation to implementation. If a method is implemented at a central main frame computer, the farmer has to pay for the time spent on optimization, and that will probably be too expensive. If it is implemented on a local personal computer or work station, very high demands on speed and internal memory must be made. These demands necessarily make the computer more expensive to purchase. Furthermore there will be a communication problem, because registrations on individual cow performance are stored in a central data base. Thus a reliable external link to the central data base is necessary (e.g. in the form of a modem). Already these hardware requirements show that *at present* a direct implementation in a decision support system is out of the question. On the other hand, it seems likely that the hardware problems will be solved in the foreseeable future, and that the local implementation probably will be preferred.

At present no software has been developed for direct implementation. The problem of transforming the mathematically expressed methods into effective computer algorithms with low time and space complexity is in no way negligible, but, on the other hand it can be done by a programming expert. The user interface should as always be given high priority, and in Denmark, the introduction of the "Integrated Farm Management System" (*Bedriftsløsningen*) at least provides a common standard. Such a standard is of great value in relation to the training and education of farmers. Thus it is concluded that the software related problems may be solved.

As mentioned, the results of this thesis are a contribution to the development of optimization techniques. The data management problem is only indirectly touched in Chapter VII, where the updating of individual traits is discussed. Until now, the parameters of the models have been estimated once for all, and the only adjustment to the individual conditions is through a herd level of milk yield which may be specified. It should be expected that parameters estimated on individual herd data would be better in the sense of less residual variance and thereby more precise predictions of the future performance of cows. However, the amount of data originating from one herd is usually too small to be used for reliable estimation of parameters.

A solution to this problem might be a more intensive application of Bayesian techniques. When the decision support system is applied the first time, a set of standard parameters are used, but as observations are done in the herd, the prior parameters are updated using Bayes theorem, so that the parameters after a few years will reflect the individual conditions of the herd. This may be regarded as a generalization of the method described in Chapter VII, where only animal specific parameters are updated. A necessary condition of using such an approach is that the problems related to organization and filtering (e.g. for outliers) are solved. A personal point of view is that the introduction of Bayesian techniques to decision support sy-

stems and monitoring will become a main issue in the research of the next decade, and if the research is successful it may lead to the breakthrough of decision support systems in animal production.

The conclusion is that an implementation of the techniques in a decision support system is not expected within the next few years. It will only be relevant when (a) the necessary hardware is available at a sufficiently low price, (b) the quality of user interfaces has increased, (c) the data management problems have been solved and (d) the educational level of farmers as concerns the use of computers has been raised. Until these conditions are met the applicational scope of the techniques is in research, as discussed in Chapter IX.







# Summary

The main purpose of this thesis is to adapt the Markov decision programming techniques to be able to cope with the animal replacement problem in a satisfactory way. The problems to be solved are the dimensionality problem (i.e. that the size of the state space tends to be so large that optimization is prohibitive), the uniformity problem (i.e. that the traits of animals are difficult to define and measure) and problems caused by herd restraints, as for instance a milk quota or a limited supply of heifers or gilts. A secondary purpose is to illustrate and discuss the applicational perspectives of the techniques.

In Chapter II a systematic survey of the developed techniques is given in the framework of traditional Markov decision programming. The notion of a hierarchic Markov process is mentioned as a way of dealing with the dimensionality problem. The uniformity problem is handled by a technique based on Bayesian updating and the herd restraints are partly solved by the introduction of a new criterion of optimality (the milk quota) and partly by a method called parameter iteration (the limited supply of heifers).

In Chapter III the notion of a hierarchic Markov process is explained. It is a series of Markov decision processes, called subprocesses, built together in one Markov decision process called the main process. The hierarchic structure is specially designed to fit replacement models, which in the traditional formulation as ordinary Markov decision processes, are usually very large. The basic theory of hierarchic Markov processes is described and examples are given of applications in replacement models. The theory can be extended to fit a situation where the replacement decision depends on the quality of the new asset available for replacement.

In Chapter IV a dairy cow replacement model based on a hierarchic Markov process is presented. In the model a cow is described in terms of lactation number, stage of lactation, the level of milk yield during the previous and present lactation, the length of the calving interval and the genetic class defined from the breeding value of the father. The

criterion of optimality is the maximization of the present value under an infinite planning horizon. Revenues from milk, calves and replaced cows, feed costs and costs of replacement heifers are considered. The future profitability calculated from the optimal solution is used for ranking of the cows in the herd. The genetic class makes it possible to include the heifers available for replacement and to let the replacement decision depend on the genetic class of the heifers.

In Chapter V a new criterion of optimality in Markov decision processes is discussed. The objective is to maximize the average net revenue per unit of physical output (or input). The criterion is relevant in some production models, where a restraint is imposed on the physical output (production quota) or on an input factor (scarce resources). An obvious application is in dairy cow replacement models under milk quotas. Iteration cycles are presented for ordinary completely ergodic Markov decision processes and for hierarchic Markov processes. The consequences of the new criterion is illustrated by a numerical example.

In Chapter VI the new criterion is applied to a dairy cow replacement model under a milk quota, and the results are compared to those under the usual discounted net revenue criterion. Optimal replacement policies, future profitabilities and rankings under the two criteria are compared. It turns out that culling should be less intensive under milk quotas because of a smaller variation in future profitability. Considerable differences in future profitability and ranking are found, and it is concluded that it is important that the correct criterion is used when milk quotas are in effect.

In Chapter VII the nature of the variation in the traits of an animal is discussed. It is argued that the variation may be described as a sum of a permanent effect which is constant over time, only varying among animals, and a temporary effect varying over time for the same animal. Only the sums of the permanent and temporary effects are observable, but we have a prior knowledge described by a probability distribution of the permanent

effect. As observations of the sums are taken, the knowledge on the true value of the permanent effect increases (i.e. the probability distribution changes). Also a more general model, involving several random traits each being influenced by several unobservable effects, is described. If the permanent effects had been directly observable, an optimal replacement policy might be determined by a hierarchic Markov process (or an ordinary Markov decision process in small models). On the other hand, the updating of knowledge on the permanent effects may be handled in a causal probabilistic net (Bayes belief net), but that method does not provide an optimization technique. Therefore, the Bayesian updating technique used in causal probabilistic nets has been combined with the optimization technique of a hierarchic Markov process in order to solve the animal replacement problem with variation in traits. The method is illustrated by a numerical example which shows that the benefits from updating of knowledge may be considerable. Furthermore, the method is compared to approaches in the literature, and it is argued that in some cases it may reduce the size of the state space in animal replacement models.

In Chapter VIII, the dairy herd is described as a multi-component system, where the components are the cows and heifers. The problem of finding an optimal replacement policy to the multi-component system is considered. The complication of the multi-component model is that if the supply of heifers is limited (i.e. the dairy farmer uses only home-grown heifers), the replacement decision concerning a cow does not only depend on the state of that particular cow, but also on the states of the other cows and heifers in the herd. Initially it is demonstrated that the multi-component replacement problem may be formulated as an ordinary Markov decision process. Unfortunately, the model is far too large to be solved by any known methods. Therefore, an approximate method combining dynamic programming and stochastic simulation in the determination of a set of descriptive parameters is suggested. The parameters are used in the calculation of the multi-component replacement criterion for cows as well as for heifers. The method has been tested by extensive simulations under 100 different conditions concerning prices and average milk yield of the herd. It is concluded that when the replacement costs (price of a heifer minus the price of a calf and the carcass value of a

cow) are small, the method improves the economic results considerably compared to the usual models, assuming unlimited supply of heifers. The information concerning heifers, which is provided by the method, makes it relevant even in cases where the replacement costs are large. The basic idea of the study may be relevant in a more general range of problems involving replacement under some constraint.

In Chapter IX, the applicational perspectives of the techniques are discussed. It is argued that they may be applied as tools in the study of the traits and conditions that influence the optimal policies and as tools in comparative studies of operational methods for application in commercial herds. The following chapters (X and XI) are examples of such applications. Direct application in commercial herds is not realistic at present.

In Chapter X, the economic value of culling information is studied. The net returns to housing, labour and management are calculated analytically using three different replacement policies in the presence and absence of a milk quota. The conclusions are that in the absence of a milk quota there are considerable benefits from using a decision support system, but under a quota the benefits are negligible if compared to the very simple policy of only replacing cows which fail to conceive within 238 days. If a system based on calculations assuming no quota is used under a quota, the dairy farmer will be directly misinformed. Decision support systems for culling should be specifically designed for the quota situation where reductions of costs are the most important means for improving herd net returns.

In Chapter XI, operational methods for direct application in commercial herds is studied. In order to develop a ranking criterion, two criteria are tested by means of stochastic simulation based on random number generation. Under the first criterion, the ranking is provided by a dynamic programming model, but instead of herd individual conditions a set of standard conditions is used. Under the second criterion the cows are ranked according to their expected net returns to housing, labour and management during the next 12 months. Both criteria are tested in the presence and absence of a milk quota. The simulation results show that both criteria are suitable for practical implementation in both situations. Thus the final choice depends on other considerations including imple-

mentation and operation costs as well as comprehensibility. A third criterion called expected maximum net returns, which has been suggested in literature, was considered, but rejected for theoretical and empirical reasons.

In Chapter XII, it is concluded that the hierarchic Markov process, as an exact and fast method, has raised the upper limit of the size of the state space to be dealt with. In that way it contributes to the solution of the dimensionality problem, but still, the upper limit has not been eliminated. The Bayesian updating technique is considered to be an important contribution to the solution

of the uniformity problem. It is expected to be useful if diseases are included in the state space. Furthermore, the method may in some cases reduce the size of the state space without loss of information. As concerns herd restraints, two methods have been proposed. A new criterion of optimality has been introduced in order to study the effect of a milk quota, and the approximate parameter iteration method has been successfully applied in a situation with shortage of heifers. As a future research area methods for state space reduction are requested.







# Dansk sammendrag

## 1. Indledning

I 1957 publicerede Bellman en bog med titlen "*Dynamic Programming*", hvori han præsenterede teorien for en ny numerisk metode til løsning af sekventielle beslutningsproblemer. De basale begreber er Bellmans optimalitetsprincip og rekursive ligningssystemer. Den grundlæggende idé kan illustreres som følger:

Antag at vi betragter et system over en endelig eller uendelig tidshorizont opdelt i perioder eller *trin*. I hvert trin observerer vi systemets *tilstand*, og vi må træffe en *beslutning* vedrørende systemet. Den valgte beslutning påvirker (deterministisk eller stokastisk) den tilstand, vi kan observere i næste trin, og afhængigt af tilstanden og den trufne beslutning modtager vi en øjeblikkelig *belønning*. Den forventede sum af alle belønninger fra og med det nuværende trin indtil slutningen af planlægningshorisonten udtrykkes ved en *værdi funktion*. Sammenhængen mellem værdifunktionens værdi i nuværende og følgende trin udtrykkes ved et sæt af rekursive ligninger. Optimale beslutninger, der afhænger af trin og tilstand, fastlægges baglæns trin for trin som de, der maksimerer højresiderne af de rekursive ligninger. En optimal *strategi* fastlagt på denne måde, opfylder kravene i Bellmans optimalitetsprincip, som lyder (i oversættelse): "En optimal strategi har den egenskab, at uanset udgangstilstanden og den hertil trufne beslutning skal de resterende beslutninger udgøre en optimal strategi med hensyn til den tilstand, der resulterer af den første beslutning" (Bellman, 1957 p. 83).

Gennem de følgende år publicerede Bellman adskillige bøger om emnet (Bellman, 1961; Bellman og Dreyfus, 1962; Bellman og Kalaba, 1965). Metoden vandt hurtigt mange fortalere, som fremførte deres synspunkter på en meget entusiastisk måde, og den forventedes at være løsningen på en lang række beslutningsproblemer fra den virkelige verden. Forhåbningerne var så store, og de fremførtes med en sådan overbevisning, at Johnston (1965) ironisk sammenlignede dynamisk programmering

med en ny religion. Samtidig var der andre, der betragtede metoden som en temmelig trivial beregningsalgoritme.

Tilsvarende beretninger kan fortælles om andre nye numeriske metoder som for eksempel lineær programmering. Som årene går, indkredsnes anvendelsesområderne dog, og oftest er konklusionen, at metoden hverken er en alt-omfattende teknik eller en trivialitet. Mellem disse ekstremer forbliver der en forholdsvis snæver gruppe af problemer, hvor metoden er et nyttigt redskab. Enten egner andre problemer sig slet ikke til metoden, eller også findes der andre overlegne måder at løse dem på.

Dette viste sig også at være tilfældet med dynamisk programmering. Et af metodens basale elementer er den sekventielle fremgangsmåde, som naturligt medfører, at den egner sig bedst til sekventielle beslutningsproblemer. *Udskiftningsproblemet* er et oplagt eksempel på et sådant problem. Når et anlæg benyttes i en produktionsproces, vil det være relevant med regelmæssige intervaller at overveje, om det nuværende anlæg bør udskiftes, eller om det med fordel kan beholdes endnu en periode. Dynamisk programmering er dermed et relevant redskab, men hvis anlæggets egenskaber er veldefinerede, og deres præcise udvikling over tiden er kendt på forhånd, vil der være analytiske metoder, som kan benyttes til forud at fastlægge det optimale udskiftningstidspunkt. Hvis på den anden side egenskaberne varierer over tid og fra anlæg til anlæg, og de yderligere udviser stokastisk variation (som det er tilfældet, når "anlægget" er et husdyr), vil beslutningen om udskiftning afhænge af de aktuelle observationer af egenskaberne. Dynamisk programmering er da en oplagt teknik til brug ved fastlæggelsen af en optimal strategi for udskiftning.

Efter denne konstatering vil vi fortsætte det historiske tilbageblik. I 1960 publicerede Howard en bog om dynamisk programmering og Markov processer. Bogens idé er at kombinere dynamisk programmering med det matematiske set velkendte begreb en "*Markov kæde*". En naturlig konsekvens heraf er at benytte betegnelsen "*Markov beslut-*

ningsproces" for det kombinerede begreb. Howard (1960) bidrog også til løsningen af problemet med optimering over et uendeligt antal trin, hvor *policy iteration* blev skabt som et alternativ til den trinvis rekursive metode, som Howard kaldte *value iteration*. Policy iteration metoden drog fordel af Markov kædens egenskaber, og der var tale om et væsentligt bidrag til udviklingen af optimeringsteknikker.

Howard udviklede metoden under to optimalitetskriterier, som var h.h.v. maksimering af de forventede totale *diskonterede* belønninger og maksimering af de forventede gennemsnitlige belønninger pr. trin. Jewell (1963) præsenterede senere en policy iteration metode til maksimering af de forventede gennemsnitlige belønninger pr. tidsenhed i semi-Markov beslutningsprocesser, hvor trinklængden er en stokastisk variabel. Howard (1971) har beskrevet en value iteration metode for sådanne processer.

Også lineær programmering blev tidligt taget i brug som en optimeringsteknik i Markov beslutningsprocesser. F.eks. har Hadley (1964) beskrevet, hvordan en optimal strategi kan bestemmes ved nævnte teknik, men så vidt vides har den aldrig været anvendt i modeller for udskiftning af husdyr. White og White (1989) har da også konkluderet, at policy iteration (undtagen i specielle tilfælde) er en mere effektiv optimeringsteknik end lineær programmering.

Siden først omtalte bog af Howard (1960) er der udført en omfattende forskning i Markov beslutningsprocesser. Der er opnået mange resultater vedrørende sammenhængen mellem de forskellige optimeringsteknikker og optimalitetskriterier. Oversigter over disse resultater kan findes hos van der Wal og Wessels (1985) så vel som hos White og White (1989).

Allerede tre år efter at Howard (1960) publicerede sin bog, offentliggjorde Jenkins og Halter (1963) en anvendelse vedrørende udskiftningsproblemet hos malkekøer. Modellen omfattede laktationsnummeret som den eneste egenskab (på 12 niveauer), og arbejdets blivende værdi var udelukkende at illustrere, at Markov beslutningsprogrammering er et muligt redskab til løsning af dette problem. Få år senere publicerede Giaever (1966) imidlertid et arbejde, som repræsenterer et virkeligt gennembrud med hensyn til anvendelse af metoden til udskiftningsproblemet hos husdyr (malkekøer). Han overvejede alle tre optimeringstek-

nikker (value iteration, policy iteration og lineær programmering), beskrev hvordan alle matematiske forudsætninger kunne opfyldes og præsenterede en fremragende model til beskrivelse af en malkekoes produktion og foderforbrug. Arbejdet har aldrig i litteraturen fået den omtale og plads, det fortjener (måske fordi afhandlingen kun er publiceret på mikrofilm). I en oversigtsartikel af van Arendonk (1984) nævnes arbejdet end ikke.

Adskillige forskere har gennem de følgende 20 år publiceret modeller for malkekoens udskiftning baseret på Markov beslutningsprogrammering, men fra et *metodemæssigt* synspunkt har ingen af disse bidraget med noget nyt sammenlignet med Giaever (1966). Adskillige studier har dog bidraget på anden måde. Smith (1971) viste, at den relativt lille model anvendt af Giaever (1966) med kun 106 tilstande på ingen måde repræsenterede den øvre grænse. Hans tilstandsrum omfattede mere end 15 000 tilstande. Kristensen og Østergaard (1982) såvel som van Arendonk (1985; 1986) og van Arendonk og Dijkhuizen (1985) undersøgte prisernes og andre forudsætningers betydning på de optimale strategier. Andre arbejder (Killen og Kearney, 1978; Reenberg, 1979) nåede næppe niveauet fra selv Jenkins og Halter (1963). Selv om udskiftningsproblemet hos søer er næsten identisk med det tilsvarende hos malkekøer, er der kun publiceret meget lidt om dette. Eneste kendte undtagelser er nye arbejder af Huirne (1990) og Jørgensen (1992).

Et arbejde af Ben-Ari et al. (1983) fortjener særlig opmærksomhed. Metodemæssigt er det ikke bemærkelsesværdigt, men det udmærker sig ved, at de væsentligste vanskeligheder i forbindelse med udskiftningsproblemet hos husdyr blev identificeret og klart formuleret. Tre forhold nævntes:

- 1) *Entydighed*. Et dyrs egenskaber er vanskelige at definere og måle. Ydermere er variationen af de enkelte egenskaber relativ stor.
- 2) *Reproduktionscyklus*. Et dyrs produktion er cyklisk. Der må træffes beslutning om *i hvilken* cyklus, der skal udskiftes såvel som *hvornår* inden for cyklus.
- 3) *Tilgang*. Der er kun et begrænset antal nye dyr (kælvekvier eller gylte) til rådighed.

Det første forhold dækker i virkeligheden over to aspekter, nemlig *entydighed*, fordi egenskaberne er

vanskelige at definere og måle, og *variabilitet*, fordi egenskaberne varierer mellem dyr og over tid. Det tredje forhold er et eksempel på en *besætningsbegrænsning*, hvormed menes en restriktion, som knytter sig til besætningen som et hele og *ikke* til det enkelte dyr. Andre eksempler på besætningsbegrænsninger er en produktionskvota eller en begrænset staldkapacitet. Vi vil derfor overveje det mere generelle problem med sådanne besætningsrestriktioner.

Konklusionen bliver således, at da forskningsaktiviteterne bag denne afhandling blev påbegyndt, repræsenterede Giaever (1966) det metodemæssige niveau, og Ben-Ari et al. havde identificeret de væsentligste vanskeligheder, som metoden bør løse. Sammenholdes Giaevers model med disse vanskeligheder, kan det konstateres, at den direkte tager højde for variabilitets-problemet, og Kristensen og Østergaard (1982) såvel som van Arendonk (1985) har senere påvist, at problemet med den cykliske produktion umiddelbart kan løses uden metodemæssige problemer. Det viser sig dog, at for at dække variabiliteten må egenskaberne repræsenteres med mange niveauer, og for at håndtere den cykliske produktion, må der indføres en ekstra tilstandsvariabel, som repræsenterer tidspunktet i cyklus. Begge forhold bidrager betydeligt til en eksplosiv vækst af tilstandsrummet. Resultatet bliver et *dimensionsproblem*. Selv om alle nødvendige betingelser for en Markov beslutningsproces er opfyldt, bliver en optimering i praksis uoverkommelig selv på moderne computere. Vanskelighederne vedrørende entydighed og besætningsbegrænsningerne blev *ikke* løst med modellen fra Giaever (1966).

Formålet med denne afhandling er at tilpasse teknikkerne for Markov beslutningsprogrammering på en sådan måde, at udskiftningsproblemet vedrørende husdyr kan behandles på tilfredsstillende vis. De vanskeligheder, der skal løses (helt eller delvist) er identificeret som dimensionsproblemet, entydighedsproblemet og problemerne vedrørende besætningsbegrænsninger. Et sekundært formål er at illustrere og diskutere de anvendelsesmæssige perspektiver af teknikkerne. Alle numeriske resultater i afhandlingen vedrører malkekøer, men Markov beslutningsprogrammering er for nyligt også taget i brug i forbindelse med søer (se f.eks. Huirne 1990 og Jørgensen 1992). Eftersom udskiftningsproblemet hos søer ikke adskiller sig meget fra det tilsvarende hos køer, vil de samme me-

todemæssige problemer melde sig, hvorfor de opnåede resultater også er relevante i so-modeller.

## 2. Sammendrag af de enkelte kapitlers resultater

Kapitel II giver en systematisk oversigt over de udviklede teknikker i relation til rammerne for traditionel Markov beslutningsprogrammering. Begrebet en hierarkisk Markov proces kan håndtere dimensionsproblemet, og en teknik baseret på bayesiansk opdatering kan behandle éntydighedsproblemet. Besætningsrestriktionerne løses dels (mælkeknoten) med introduktion af et nyt optimalitetskriterium og dels (begrænset tilgang af kælvekvier) med en metode kaldet parameter iteration.

Kapitel III forklarer begrebet hierarkiske Markov processer. Der er tale om en serie af Markov beslutningsprocesser kaldet underprocesser, som bygges sammen i én Markov beslutningsproces, der betegnes som hovedprocessen. Den hierarkiske struktur er specielt designet med henblik på udskiftningsmodeller, der i traditionel formulering som almindelige Markov beslutningsprocesser normalt er meget store. Kapitlet beskriver den basale teori for hierarkiske Markov processer, og giver eksempler på anvendelse i udskiftningsmodeller. Teorien kan udvides til at dække en situation, hvor beslutningen om udskiftning afhænger af kvaliteten af det nye aktiv, som overvejes indsat i stedet for det nuværende.

Kapitel IV præsenterer en model for malkekoens udskiftning baseret på en hierarkisk Markov proces. Modellen beskriver en ko ved dens laktationsnummer, laktationsstadium, mælkeydelsen i foregående og nuværende laktation, kælvningsintervallets længde og det genetiske niveau udtrykt ved faderens avlsværdi. Optimalitetskriteriet er maksimering af nutidsværdien under uendelig planlægningshorisont. Indtægter fra mælk, kalve og udsæterkøer samt udgifter til foder og kælvkvier tages i betragtning. Den fremtidige profitabilitet beregnet ud fra den optimale løsning anvendes til indbyrdes rangering af køerne i en besætning. Det genetiske niveau gør det muligt at inddrage kælvkvierne i denne rangering og således lade udskiftningsbeslutningen afhænge af kælvkviernes afstamning.

Kapitel V diskuterer et nyt optimalitetskriterium til brug i en Markov beslutningsproces. Målet er at maksimere det gennemsnitlige nettoudbytte pr. produkt- eller faktorenhed. Kriteriet er relevant i forbindelse med visse produktionsmodeller, hvor der er en begrænsning på produktionen (kvota) eller på en faktor (knap ressource). Et oplagt anvendelsesområde er i udskiftningsmodeller for malkekoen under en mælkekquota. Der præsenteres optimeringscykler for almindelige Markov beslutningsprocesser såvel som for hierarkiske Markov processer. Et numerisk eksempel illustrerer konsekvenserne af at anvende det nye kriterium.

Kapitel VI anvender det nye kriterium i en udskiftningsmodel for malkekoen under en mælkekquota, og sammenligner resultaterne med de tilsvarende under det sædvanlige diskonterede nettoudbytte-kriterium. Optimale strategier, fremtidige profitabiliteter og rangeringer sammenlignes. Det viser sig, at udskiftningen bør være mindre intensiv under en mælkekquota på grund af en noget mindre variation i den fremtidige profitabilitet. Betydelige forskelle konstateres også i den indbyrdes rangering, og det konkluderes, at det er væsentligt, at det korrekte kriterium anvendes ved produktion under en kvota.

Kapitel VII diskuterer selve naturen af den konstaterede variabilitet i dyrs egenskaber og argumenterer for, at variationen kan beskrives som en sum af en permanent del, som er konstant over tid og kun varierer mellem dyr, og en midlertidig effekt, som varierer over tiden for det samme dyr. Kun summen af den permanente og midlertidige påvirkning kan observeres, men der vil typisk være en *a priori* viden i form af en sandsynlighedsfordeling for den permanente påvirkning. Regelmæssige observationer af summen øger den til rådighed værende viden om den permanente påvirkning (d.v.s. ændrer sandsynlighedsfordelingen). Kapitlet beskriver også en mere generel model med observation af flere egenskaber, som hver især er under indflydelse af flere ikke-observerbare påvirkninger. Hvis den permanente påvirkning var direkte observerbar, kunne en optimal strategi uden videre bestemmes ved hjælp af en hierarkisk Markov proces (eller en traditionel Markov beslutningsproces i små modeller). På den anden side kan opdateringen af den foreliggende viden om den permanente påvirkning håndteres i et kausalt probabilistisk net, men i så fald rådes der ikke over en optimeringsteknik. Derfor kombineres den bayesianske opda-

teringsteknik med optimeringsteknikken for hierarkiske Markov processer med henblik på at løse udskiftningsproblemet vedrørende husdyr, hvor egenskabernes manglende entydighed er et problem. Et numerisk eksempel illustrerer metoden og viser, at gevinsten ved opdatering kan være betydelig. Ydermere sammenlignes metoden med anvendte fremgangsmåder i litteraturen, og der argumenteres for, at i visse tilfælde kan den medvirke til reduktion af tilstandsrummet uden tab af information.

Kapitel VIII beskriver en kvægbesætning som et multi-komponent system, hvor de enkelte komponenter er kørerne og kvierne. Problemet med fastlæggelsen af en optimal udskiftningsstrategi i et sådant system overvejes. Komplikationen består i, at hvis tilgangen af kælvkvier er begrænset (når mælkeproducenten kun anvender egne kvier), afhænger udskiftningsbeslutningen for en ko ikke blot af dennes egne egenskaber, men også af de øvrige køers og kviers egenskaber. Indledningsvis påvises det, at multi-komponent problemet principielt kan beskrives som en sædvanlig Markov beslutningsproces. Desværre er modellen dog alt for stor til at en optimal strategi kan bestemmes med nogen kendt teknik. Kapitlet præsenterer derfor en approksimativ metode, som kombinerer Markov beslutningsprogrammering og stokastisk simulering til bestemmelse af et sæt beskrivende parametre. Parametrene bliver brugt til fastlæggelse af et multi-komponent udskiftningskriterium for malkekørerne såvel som kvierne. Metoden er testet med omfattende simuleringer under 100 forskellige betingelser vedrørende priser og gennemsnitsydelse. Det konkluderes, at når de direkte udskiftningsomkostninger (prisen på en kælvkvie minus prisen for en kalv og prisen på en ung udsætterko) er små, forbedrer metoden det økonomiske resultat betydeligt sammenlignet med sædvanlige metoder, hvor ubegrænset tilgang af kælvkvier forudsættes. Metodens afledte informationer vedrørende kvierne er relevant selv i en situation, hvor udskiftningsomkostningerne er store. Metodens basale idé kan være relevant for en mere generel kreds af problemer, som involverer udskiftning under en eller anden form for besætningsbegrænsning.

Kapitel IX diskuterer de anvendelsesmæssige perspektiver af de udviklede teknikker og argumenterer for, at de kan finde anvendelse i undersøgelser af hvilke egenskaber og betingelser, der påvirker de optimale strategier samt i sammenlignende studier af operationelle metoder til direkte an-

vendelse i praksis. De følgende kapitler (X og XI) er eksempler på sådanne anvendelser. Direkte anvendelse af teknikkerne er ikke realistisk på nuværende tidspunkt.

Kapitel X undersøger den økonomiske værdi af udskiftningsinformation. Restbeløb til stald, arbejde og driftsledelse beregnes analytisk under tre forskellige udskiftningsstrategier med og uden mælkekvote. Det konkluderes, at uden mælkekvote vil nytteværdien af et beslutningsstøtte system være betydelig, men under kvotering er nytteværdien forsvindende sammenlignet med blot at følge en strategi med udskiftning af alle køer, som ikke bliver drægtige inden for de første 238 dage efter kælvning. Beslutningsstøtte systemer for udskiftning må designes specielt til kvota-situationen, hvor reducerede omkostninger er det mest betydningsfulde middel til forøgelse af det samlede restbeløb. Hvis et beslutningsstøtte system ignorerer betydningen af en mælkekvote, vil mælkeproducenten direkte blive misinformeret.

Kapitel XI sammenligner operationelle metoder til direkte anvendelse i praksis. Med sigte på udvikling af et rangeringskriterium, testes to metoder ved hjælp af stokastisk simulering med generering af pseudo-tilfældige tal. Ved den ene metode frembringes rangeringen ved Markov beslutningsprogrammering, men i stedet for besætningspecifikke forudsætninger anvendes et sæt af standardforudsætninger. Ved den anden metode rangeres køerne i henhold til deres forventede restbeløb til stald, arbejde og driftsledelse gennem de følgende 12 måneder. Begge metoder er testet med og uden mælkekquotering. Simuleringsresultaterne viser, at begge metoder er egnede til praktisk anvendelse i begge situationer. Det endelige valg vil derfor afhænge af andre overvejelser, herunder gennemskueligheden samt omkostningerne ved implementering og drift. En tredje metode baseret på begrebet "forventet maksimalt restbeløb" (foreslået i litteraturen) blev overvejet, men af teoretiske og empiriske årsager blev den forkastet.

### 3. Konklusioner og videre perspektiver

Indledningen angav, at det primære formål var at tilpasse teknikkerne for Markov beslutningsprogrammering på en sådan måde, at udskiftningsproblemet vedrørende husdyr kunne behandles på en

tilfredsstillende måde. Det kan nu konstateres, at de tre identificerede vanskeligheder i denne forbindelse *dimensionsproblemet*, *entydighedsproblemet* og problemerne vedrørende *besætningsrestriktioner* indbyrdes påvirker hinanden. Som påvist i kapitel VII kan løsningen af entydighedsproblemet også i visse tilfælde medvirke til løsning af dimensionsproblemet gennem reduktion af tilstandsrummet, og som det fremgår af kapitel VIII, kan besætningsrestriktioner gøre dimensionsproblemet værre, idet alle dyr må vurderes simultant. Derfor vil løsning af problemerne vedrørende besætningsrestriktioner også i nogle modeller bidrage til løsning af dimensionsproblemet.

Det væsentligste bidrag til løsning af dimensionsproblemet er dog formuleringen af hierarkiske Markov processer. De beregningsmæssige fordele ved teknikken er illustreret ved teoretiske overvejelser samt et numerisk eksempel i kapitel II. De væsentligste fordele ved teknikken kan sammenfattes således:

- 1) Metoden er eksakt.
- 2) Metoden er, som det fremgår af kapitel II, langt hurtigere end value iteration (d.v.s. den traditionelle dynamisk programmerings teknik).
- 3) Metoden gør det, som det fremgår af kapitel X, muligt direkte at beregne konsekvenserne af en vilkårlig strategi. Ydermere er det muligt direkte at beregne en lang række tekniske og økonomiske resultater under en given strategi som beskrevet i flere kapitler (f.eks. årlig udskiftningsprocent, årlig mælkeydelse, gennemsnitlig kuldstørrelse etc.).

Eftersom metoden er hurtigere end sædvanlige teknikker, *bidrager* den til løsning af dimensionsproblemet, fordi optimering af langt større modeller bliver realistisk med et overkommeligt tidsforbrug. På den anden side *løses* problemet ikke. Grænserne for det mulige hæves, men fjernes ikke.

Med hensyn til entydighedsproblemet, foreslås den bayesianske teknik fra kapitel VII. Teknikkens styrke er, at den opstiller en ramme for håndtering af mangelfuld viden. Ved et hvilket som helst niveau af viden er det muligt at vælge en optimal beslutning, *under de foreliggende omstændigheder* (med mangelfuld viden). Den bayesianske fremgangsmåde synes derfor at være i bedre overensstemmelse med virkelighedens vilkår. Det forven-

tes, at metoden især vil være relevant i forbindelse med inddragelse af kategoriske egenskaber som sygdomme i beslutningsgrundlaget som skitseret i kapitel VII. Dette perspektiv må dog afvente test på et realistisk datagrundlag, før der kan drages nogen endelig konklusion. Med hensyn til anvendelse i forbindelse med sædvanlige kvantitative egenskaber som mælkeydelse og kuldstørrelse forventes metoden at kunne blive et væsentligt bidrag til løsning af entydighedsproblemet, fordi den gør det muligt at skelne mellem variation forårsaget af forskellige kilder, og således kan tage højde for selve variationens natur. Ydermere kan metoden i visse tilfælde medvirke til reduktion af tilstandsrummet uden tab af information.

To forskellige besætningsbegrænsninger er overvejet i denne afhandling. Den ene begrænsning er en mælkekvota, som der er taget højde for ved indførelse af et alternativt optimalitetskriterium, hvor det gennemsnitlige nettoudbytte pr. kg. mælk maksimeres (kapitel V og VI). Det nye kriterium løser det mere langsigtede rangeringsproblem for en besætning, der producerer mælk under kvotering, men det løser *ikke* det helt kortsigtede problem med hensyn til på et vilkårligt tidspunkt at kunne beslutte den optimale besætningsstørrelse. For at kunne løse også dette problem vil det være nødvendigt at ty til parameter iterations metoden, som i kapitel VIII er anvendt i forbindelse med en anden begrænsning på besætningsniveau, nemlig en begrænset tilgang af kælvekvier. Det er påvist, at i en situation med mangel på kælvekvier er metoden overlegen i forhold til anvendelse af sædvanlige enkelt-dyr modeller, men der er ikke tale om en alt omfattende teknik, som uden videre kan overføres direkte til andre typer af besætningsbegrænsninger. Den grundlæggende idé er at drage nytte af det faktum, at en optimal strategi for problemet uden restriktion allerede er kendt og derefter at overveje, hvordan begrænsningen logisk set må påvirke den totale nutidsværdi (eller relative værdi) for hele besætningen. Disse *grundtræk* forventes at kunne anvendes også under andre besætningsbegrænsninger, men i hvert enkelt tilfælde resterer stadig et møjsommeligt arbejde med identifikation af relevante funktioner og parametre. Således kan problemerne vedrørende besætningsbegrænsninger ikke siges at være *løst*, men der er opstillet nogle *rammer*, som i det konkrete tilfælde kan danne udgangspunkt for en løsning.

Et sekundært formål med denne afhandling er at

illustrere og diskutere de anvendelsesmæssige perspektiver for de udviklede teknikker. I kapitel IX identificeredes anvendelsesområderne som værende *forskning, udvikling af operationelle metoder for praktisk anvendelse og direkte anvendelse i praksis*. Som illustreret i kapitlerne X og XI kan det konkluderes, at teknikkerne er nyttige redskaber i den forskning, der studerer de betingelser og egenskaber, som påvirker de optimale strategier. Yderligere finder de anvendelse i komparative studier, hvor operationelle metoder udvikles til praktisk anvendelse (jvf. kapitel XI). Hvad angår direkte anvendelse i praksis, må det antages, at dette teknisk set vil blive muligt. Hvorvidt det også er hensigtsmæssigt, afhænger af resultaterne af sådanne komparative studier. På nuværende tidspunkt kan der ikke drages nogen endelig konklusion på dette punkt.

I de fremtidige forskningsaktiviteter på området, bør dimensionsproblemet gives høj prioritet. To alternative veje kan vælges. Ad den ene vej arbejdes der videre på yderligere at hæve grænsen for det mulige med hensyn til tilstandsrummets størrelse. Den dag, hvor en multi-komponent model som den i kapitel VIII beskrevne kan løses eksakt, er dog meget fjern, medmindre der gøres en virkelig epokegørende opdagelse. Yderligere er det et spørgsmål, om ikke grænsen for den menneskelige evne til at fatte og overskue modeller af en sådan størrelse overskrides forinden. En anden vej kan vælges: At udvikle metoder til transformation af tilstandsrummet, så størrelsen begrænses uden (eller med kun lille) tab af information. En yderligere udbygning af de bayesianske teknikker kunne være en mulighed, men langt mere forskning på dette område vil være ønskelig.

Også spørgsmålet om, hvordan oplysninger om sundhedsstatus kan inddrages i beslutningsgrundlaget, bør studeres nærmere. Igen er de bayesianske teknikker en mulighed. Det forekommer også oplagt at problemet med besætningsbegrænsninger må være et væsentligt element i fremtiden, eftersom husdyrproduktion i den virkelige verden finder sted i besætninger. Forestillingen om en simpel sammenligning af et enkelt dyr med en mulig kælvekvie/gylt er uholdbar i en situation, hvor alle dyrene konkurrerer om den samme knappe ressource eller produktionskvota.

De nævnte fremtidige forskningsområder viser, at der stadig er behov for en indsats på det metodiske område, hvis formålet med dette arbejde skulle

opfyldes fuldstændigt. Det skal dog bemærkes, at *selv om* alle metodiske problemer blev løst, og følgelig det teknisk definerede formål opfyldtes, var arbejdet stadig ikke fuldført. Det forekommer naturligt også at overveje det i et lidt videre perspektiv. På længere sigt er det naturligvis tanken, at de udviklede teknikker skal kunne støtte husdyrbrugeren i dennes konkrete beslutninger om hvilke dyr, der skal udsættes. Med andre ord kunne teknikkerne udgøre den centrale del af et beslutningsstøtte system.

Ingen af de udviklede metoder er på nuværende tidspunkt implementeret i noget beslutningsstøtte system, og det er derfor naturligt at overveje, hvad årsagen kan være. Situationen er på ingen måde unik. På trods af mindst to årtiers forskning i optimerings- og simuleringss modeller er så godt som intet nået ud til husdyrbrugeren i form af fungerende beslutningsstøtte systemer. En international kongres for computer teknologi i 1990 var tænkt som en præsentation af vellykkede praktiske anvendelser. Efter kongressen udtalte en deltager ironisk, at nogle af anvendelserne ganske rigtigt var vellykkede, medens andre var praktiske, og atter andre var blot anvendelser. Med andre ord synes vellykkede praktiske anvendelser at være undtagelser. Titlen på den følgende kongres i samme serie kan oversættes til "Computer-teknologi søger brugere", og antyder, at dette er den generelle opfattelse.

Den generelle mangel på vellykkede beslutningsstøtte systemer i husdyrproduktionen gør det naturligt at overveje, hvad der i almindelighed betinger succesen af et sådant system. Det følgende vil være en gennemgang af *nødvendige* betingelser (flaskehalse) for succes. (Der er således ikke nødvendigvis tale om *tilstrækkelige* betingelser). Der vil blive skelnet mellem om betingelserne vedrører *hardware*, *software*, *metode* eller de knytter sig til *husdyrbrugeren*.

Kravene til hardware vil til dels gå på egenskaber som hurtighed, disk plads og intern hukommelse, idet nogle metoder kræver endog meget kraftige maskiner. På *kort sigt* kan disse egenskaber tænkes at være flaskehalse i udviklingen af beslutningsstøtte systemer, men som illustreret i kapitel IX forbedres disse specifikationer meget hurtigt i takt med introduktion af nye modeller, så på *langt sigt* vil det næppe være på dette felt, der opstår problemer. En anden flaskehals på hardware området er kommunikationsudstyr, der gør det muligt at

modtage eksterne data om f.eks. klassificering af dyr og mælk fra slagteri og mejeri såvel som interne data som f.eks. temperatur og konduktivitet af mælk eller slagtekyllingers vægt fra sensorer. Der kan også være behov for at sende data fra computeren til automatisk udstyr som malkerobotter og foderautomater. Også på dette område går udviklingen dog meget hurtig, så på længere sigt vil der næppe heller opstå problemer på dette område. Det er dog for hele hardware/software området en betingelse, at prisen er så lav, at en investering i udstyret er rentabel.

På software området må der dels udvikles de nødvendige applikationer, hvorved forstås programmering af optimeringsmetoder og datahåndtering, og dels let forståelige brugerflader. Især betydningen af en god brugerflade er ofte blevet overset, fordi der er fokuseret på programmering af de udviklede metoder. Ikke desto mindre vil et ellers godt beslutningsstøtte system næppe have nogen chance på markedet, hvis ikke brugerfladen er hensigtsmæssig. Et højt niveau i software udviklingen sikres formentlig bedst gennem en erkendelse af, at denne del af arbejdet bør overlades til specialister i programmering (f.eks. ansat af rådgivningstjenesten). Forskeren, der udviklede metoden, er normalt ikke ekspert i programmering, og ydermere er han eller hun langt fra uvildig i vurderingen af de anvendelsesmæssige perspektiver. På den anden side kræver en hurtig implementering af nye metoder, at der er en tæt kontakt mellem forskere og rådgivningstjeneste.

Inden for forskningen er hovedvægten tilsyneladende blevet lagt på udvikling af *metoder*. Det gælder især "optimeringsmetoder" i en bred betydning, der også dækker eksempelvis ekspertsystemer og simulering. Forskerne har udviklet adskillige metoder og prototyper og har normalt ikke været tilbageholdende med selv at prise disses muligheder i praksis, men som nævnt har meget lidt nået husdyrbrugeren. Eftersom nytteværdien af metoderne i flere tilfælde har været dokumenteret, kan det konkluderes, at mangel på optimeringsmetoder næppe er en flaskehals i udbredelsen af beslutningsstøtte systemer. Lige så vigtig (men mindre udforsket) er datahåndteringen. Som resultat af den teknologiske udvikling antages antallet af data kilder at stige. Sensorer og eksterne kommunikationslinjer forventes at kunne bidrage med hidtil usete mængder af registreringer. Bearbejdningen af disse registreringer til data og informationer kræ-

ver udvikling af nye metoder til opgaver som filtrering, organisering af data baser og estimation af parametre på besætningsniveau.

Sidst, men ikke mindst, bør *husdyrbrugerens* situation overvejes. Er produktionen stor nok til at retfærdiggøre købet af et beslutningsstøtte system, og tillader økonomien det? Selv hvis svaret er ja, kan husdyrbrugerens uddannelse og manglende erfaring med computere være et problem. Dette er nok den mest begrænsende faktor i udbredelsen af beslutningsstøtte systemer. Manglende computererfaring kan delvist afhjælpes ved efteruddannelse og gode brugerflader, men det må erkendes, at der kan gå en generation, før gennemsnits producenten er virkelig fortrolig med computere og opfatter dem som lige så naturlige redskaber som papir og blyant. Uddannelse er også væsentlig i en bredere betydning end blot brug af computere. For eksempel kan manglende kendskab til statistisk teori være et problem i relation til stokastisk modellering, konfidensgrænser, følsomheds overvejelser etc.

Denne korte gennemgang af nødvendige betingelser for et vellykket beslutningsstøtte system synes at vise, at det ikke er overraskende, at de er så få. Hovedproblemet er, at der har været en omfattende forskning i metodeudvikling uden at niveauet af software (herunder især brugerflader) og uddannelse er fulgt med.

Efter disse generelle overvejelser er det naturligt at vende tilbage til de konkrete teknikker, der præsenteres i denne afhandling, og overveje hvilke problemer der ville opstå ved implementering i et beslutningsstøtte system. Som ved alle andre eksempler i de foregående kapitler tages der udgangspunkt i en malkekvægsbesætning under danske forhold.

De udviklede metoder stiller store krav til computerens regnekraft. Allerede dette forhold kunne give anledning til problemer i forbindelse med en implementering. Hvis det implementeres på et centralt anlæg, må mælkeproducenten betale for tidsforbruget, hvilket antageligt vil være for dyrt. Vælges en decentral løsning stiller det meget store krav til regnehastighed og intern hukommelse, hvilket uundgåeligt gør anskaffelsen dyr. Desuden vil der være et kommunikationsproblem, fordi oplysningerne om de enkelte kørs præstationer er lagret centralt i kvægdatabasen på Landbrugets EDB Central. En stabil forbindelse (f.eks. i form af et modem) til denne database vil således være nød-

vendig. Alene disse hardware behov viser, at *på nuværende tidspunkt* er en direkte implementering i et beslutningsstøtte system udelukket. På den anden side forekommer det sandsynligt, at hardware problemerne vil blive løst inden for en overskuelig fremtid, og at den decentrale løsning i så fald vil være at foretrække.

På nuværende tidspunkt findes heller intet software til direkte implementering. Det er på ingen måde en banal opgave at omsætte de matematisk formulerede metoder til effektive computer algoritmer uden overflødig tids- og lagerforbrug, men på den anden side vil det naturligvis være muligt for en dygtig programmør. Som altid bør brugerfladen prioriteres højt, og fremkomsten af "Bedriftsløsningen" i Danmark gør, at der i det mindste findes en fælles standard, hvilket er af stor værdi i forbindelse med oplæring og uddannelse af mælkeproducenten. Det kan derfor konkluderes, at det vil være teknisk muligt at fremstille det nødvendige software.

Som nævnt repræsenterer denne afhandling et bidrag til udvikling af optimeringsteknikker. Datahåndteringsproblemet berøres kun indirekte i kapitel VII, hvor opdatering af viden om det enkelte dyrs egenskaber diskuteres. Som modellerne foreligger nu, er alle parametre estimeret én gang for alle, og den eneste tilpasning til den enkelte besætning er angivelse af et besætningsniveau for mælkeydelse. Det må forventes, at parametre estimeret på besætningsniveau vil være bedre med mindre residualvarians og dermed mere præcise prædiktioner til følge. Imidlertid er mængden af data fra en enkelt besætning normalt for lille til en præcis estimation af parametre.

En løsning af dette problem kunne være en mere intensiv brug af Bayesianske metoder. Når beslutningsstøtte systemet først tages i brug, anvendes et sæt af standard parametre, men efterhånden som der gøres observationer i besætningen opdateres de oprindelige parametre ved hjælp af Bayes' regel, så de efter nogle få år vil afspejle forholdene i den enkelte besætning. En sådan fremgangsmåde kan betragtes som en generalisering af metoden beskrevet i kapitel VII, hvor dog kun parametre relateret til enkelte dyr opdateredes. En nødvendig forudsætning for at gøre brug af en sådan fremgangsmåde er, at problemerne vedrørende data organisering og filtrering løses. Et personligt synspunkt er, at inddragelse af Bayesianske metoder i beslutningsstøtte systemer vil blive et hovedområde i det

næste årtis forskning, og hvis det lykkes, kan det føre til det endelige gennembrud for sådanne systemer i husdyrbruget.

Konklusionen er, at en implementering af de præsenterede teknikker i et beslutningsstøtte system ikke kan forventes inden for de nærmeste få år. Det vil først være relevant, når (a) det nødvendige hardware er tilgængeligt til en rimelig pris, (b) kvaliteten af brugerflader i almindelighed er forbedret, (c) datahåndteringsproblemet er løst og

(d) husdyrbrugernes uddannelsesniveau vedrørende brug af computere er forbedret. Indtil disse betingelser er opfyldt, vil anvendelserne være begrænset til forskningen som beskrevet i kapitel IX.

## Referencer

Der henvises til litteraturlisten hørende til kapitel I.



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**Markov decision programming  
techniques applied to the  
animal replacement problem**

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